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Abstract

While there are now a number of empirical studies on the subject, very little is known on the market price for default risk from a theoretical perspective. This paper is a first step in the direction of an equilibrium model for the pricing of defaultable securities in an incomplete market setup. We first provide an explicit characterization of the set of equivalent martingale measures consistent with no arbitrage in the presence of default risk, as well as a necessary and sufficient condition for a convenient separation between adjustments for market risk and default risk. That result allows us to spell out an unambiguous definition of the market price for default risk as the logarithm of the ratio of the risk-adjusted probability of default to the original probability of default. It also suggests the following question: how should the original probability of default be adjusted to account for agents' risk-aversion? We address this question in a dynamic continuous-time equilibrium setup, and obtain a defaultable version of a standard consumption-based capital asset pricing model. In particular, we confirm the intuition that the correlation between default risk and market risk is a key ingredient of the equilibrium price for default risk, and obtain a quantitative estimate of the magnitude of the effect. Our model is consistent with empirical findings in that it predicts that the term structure of credit spreads can be upward sloping with a non-zero intercept. The theory is illustrated by an application to the valuation of employee compensation packages, which may be regarded as peculiar, yet natural, examples of defaultable securities.

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Abstract

While there are now a number of empirical studies on the subject, very little is known on the market price for default risk from a theoretical perspective. This paper is a first step in the direction of an equilibrium model for the pricing of defaultable securities in an incomplete market setup. We first provide an explicit characterization of the set of equivalent martingale measures consistent with no arbitrage in the presence of default risk, as well as a necessary and sufficient condition for a convenient separation between adjustments for market risk and default risk. That result allows us to spell out an unambiguous definition of the market price for default risk as the logarithm of the ratio of the risk-adjusted probability of default to the original probability of default. It also suggests the following question: how should the original probability of default be adjusted to account for agents' risk-aversion? We address this question in a dynamic continuous-time equilibrium setup, and obtain a defaultable version of a standard consumption-based capital asset pricing model. In particular, we confirm the intuition that the correlation between default risk and market risk is a key ingredient of the equilibrium price for default risk, and obtain a quantitative estimate of the magnitude of the effect. Our model is consistent with empirical findings in that it predicts that the term structure of credit spreads can be upward sloping with a non-zero intercept. The theory is illustrated by an application to the valuation of employee compensation packages, which may be regarded as peculiar, yet natural, examples of defaultable securities.

Credit risk has recently attracted much attention because of a dramatic increase in the severity and frequency of losses arising from default. While few issuers of speculative bonds defaulted on their obligations to creditors when the market was at its infancy, even during the severe recessions of 1980-1982, the default rate on speculative-grade bonds has significantly increased in the more recent past, and even soared to 11% in 1990-1991 (Helwege and Kleiman (1997)). In the meantime, the interest for credit risky securities has also increased, and the market for high-yield or speculative-grade bonds has grown from $30 billion of outstanding bonds in 1980 to nearly $250 billion today. Despite the existence of a number of empirical studies on the subject (see in particular Duffee (1999) or Elton at al. (2000) for recent references), very little is known, however, on the *market price for default risk* from a theoretical standpoint. The mere absence in the literature of a clear definition of that concept is perhaps the best evidence that a good understanding of how investors implicitly risk-adjust the probability of default as they set equilibrium prices is currently missing[1].

This is perhaps surprising, given that the need for a better understanding of the nature of default risk has now surged a rich literature on the pricing of defaultable securities. The models introduced in the credit risk literature may be divided into two categories: models in which default is based on the value of the firm (also known as *structural models* or *contingent-claim models*) and *reduced form models* (also known as *intensity-based models*). The first example of a structural model of defaultable bonds goes back to seminal papers by Black and Scholes (1973) and Merton (1974), based on the observation that the equity of a firm may be regarded as a call option on the value of the firm. Subsequently, many authors have attempted to relax some of the restrictive assumptions of Merton's (1974) model. In particular, Black and Cox (1976) relax the restrictive assumption that a default can occur only at maturity, and introduce a more general default that can occur at any date during the bond lifetime. As in most structural models, the date of default is modelled as the first hitting time of a given barrier by a process describing the value of the assets of the firm[2]. A distinctive feature is that default does not come as a surprise to an agent who observes asset prices. A consequence is that there is no real timing risk involved, or more precisely that timing risk is embedded within asset price risk, so that a specific inspection of the market price for default risk is neither desirable, nor even conceivable; default risk actually does not exist per se. In this case, markets actually remain complete, and arbitrage valuation of risky debt can be performed in a way initiated by Merton (1974). The assumption of complete markets, however, may

[1] We provide an unambiguous definition of the market price for default risk in this paper (see Section 2).

[2] More restrictive assumptions in Merton's model have been relaxed in some subsequent models, such as Longstaff and Schwartz (1995). In particular they allow for interest rate risk and for complex capital structure and priority rules. By introducing bankruptcy costs and tax effects, this framework has further been extended to encompass endogenous default, optimally triggered by equity owners (see for example Leland (1994, 1998), Leland and Toft (1996) and Mella-Baral and Perraudin (1997)).

not be satisfied in practice. In Duffie and Lando (2000), for example, some incompleteness is induced by the fact that agents have incomplete information about the value of the assets of the firm, due to imperfect accounting reports. Various reduced-form models have been introduced in an attempt to provide insights into the pricing of defaultable securities in the presence of incomplete markets. Important examples are Duffie, Schroder and Skiadas (1996), Duffie and Singleton (1999), Jarrow and Turnbull (1992, 1995), Lando (1997), or Madan and Unal (1994), among others. The main result in that literature is that, in the absence of arbitrage, the value of a promised payoff subject to default risk is similar to the value of an otherwise identical default-free payoff, discounted with a suitably adjusted risk-free rate equal to the actual risk-free rate plus the instantaneous probability of default.

Since these papers have essentially focused on discussing the implication of the absence of arbitrage on the prices of defaultable securities, they fail to provide any insight into the market price for default risk. More specifically, while most authors acknowledge that *some* risk-adjusted probability of default, and not the original probability of default, should be used in the pricing formulas, nothing explicit is said about what that adjustment should be in a standard economy. Being able to map an original probability of default into a risk-adjusted probability of default is, however, needed for asset pricing purposes. Conversely, using prices of defaultable securities or credit derivatives, one may obtain an implicit value for the equilibrium risk-adjusted probability of default, and then map it back into an original probability of default, to be used, for example, for risk management purposes.

It is precisely the focus of this paper to improve our understanding of that adjustment, that is understanding which is *the* risk-adjusted probability of default, among the infinitely many different choices consistent with the absence of arbitrage, that is implicitly used by agents when they set equilibrium prices of defaultable securities. As such, our paper can be regarded as completing the literature on default risk by providing a first step in the direction of an equilibrium model for the pricing of defaultable securities in an incomplete market setup. More specifically, we first provide an explicit characterization of the set of equivalent martingale measures in the presence of default risk (proposition 2), as well as a necessary and sufficient condition for a convenient separation between adjustments for market risk and default risk (equation (3)). That result allows us to spell out an unambiguous definition of the market price for default risk as the logarithm of the ratio of the risk-adjusted probability of default to the original probability of default (definition 1). We regard this as a first contribution of the paper. It also suggests the following question: how should the original probability of default be adjusted to account for agents' risk-aversion? We address this question in a dynamic continuous-time equilibrium setup, and we provide a closed-form solution for the price of a unit defaultable bond under specific assumptions. We also derive an explicit expression for the market price of default risk, which conveniently allows one to map an original probability of default into a risk-adjusted probability of default. We consider that result as the main contribution of the paper. We show in particular that the default risk premium is a linear function of the market risk premium (see equation (12)). This is consistent with Jagannathan and Wang (1996), who *assume* that the equity premium is a linear function of the default premium in the economy, and also with Chen, Roll and Ross (1986), who provide evidence that the spread on high-yield bonds explain the returns on stocks. In that respect, our results provide new insights into the understanding of the relationship market risk and default risk; we confirm the intuition that the correlation between default risk and market risk is a key ingredient of the equilibrium price for default risk, and obtain a quantitative estimate of the magnitude of the effect. Our model also predicts that the term structure of credit spreads can be upward sloping with a non-zero intercept, which is consistent with empirical findings (Helwege and Turner (1999)).

Upon completion of this paper, we became aware of a related approach by Jarrow, Lando and Yu (2000), who provide conditions under which the *idiosyncratic* component of default risk may not priced, using approximate and exact notions of conditional diversification. Their conclusions and ours are consistent in the sense that both papers acknowledge that some adjustment to the actual probability of default is to be expected because of the presence of a *systematic* component of default risk. Our paper complements that paper in the following way. While Jarrow, Lando and Yu (2000) investigate the question of the market price for default risk in an APT type of framework, we use instead an equilibrium model. One main difference is that, by imposing a specific structure to the economy (a standard CCAPM framework), we are able to provide an explicit expression for the market price for default risk, a result that cannot be obtained under an APT framework. Another attempt to cast the default risk literature in an equilibrium setting can be found in a recent paper by Chang and Sundaresan (1999). The main difference with our paper is that they consider an endogenous timing of default. In a nutshell, they are providing the equilibrium counterpart to structural models of default, while we are providing the equilibrium counterpart to reduced-form models of default.

These results are illustrated through an application to the valuation of employee compensation packages, which may be regarded as peculiar, yet natural, examples of defaultable securities. Timing risk there translates into uncertainty over vesting. If for some reason, voluntary or involuntary, an employee leaves the company before the vesting date, then the promised compensation package is not delivered; it essentially goes into default. The question of how to account for the probability of *forfeiture* has not received a proper treatment in the literature. For example, a simple rule has been recommended by the Financial Accounting Standards Board (FASB)[3], which consists in multiplying the value of an otherwise identical

[3] See FASB exposure draft 127-C (1993).

package by the probability that the employee is still with the company at the vesting date. The problem here is that, by taking a simple expectation under the true probability measure, risk-neutrality with respect to vesting risk, or equivalently a zero price for vesting risk, is implicitly assumed. A similar zero-price assumption is also widely used in the literature (e.g. Jennergren and Naslund (1992), Kulatilaka and Marcus (1994), Rubinstein (1995)). In general, however, one expects the value of a compensation package to be given by the value of an otherwise identical default-free package multiplied by a *risk-adjusted* probability of non-vesting. Without an analysis such as the one developed in this paper, what this adjustment should be is not transparent. That there is no readily available answer to this question has been for example noted by Rubinstein (1995): *"Simply multiplying by one minus the probability of forfeiture, as proposed by FASB, presupposes that the market discounts the uncertainty associated with forfeiture as if it were risk-neutral toward this risk. In fact, for the reasons explained above*[4]*, this risk is likely to be negatively correlated with the value of a well-diversified portfolio, and its effect on valuation should be handled using risk-adjusted discounting – a serious complication about which the theory of finance has no easy answers."* It is our hope that the present paper provides some simple and useful insights into that "serious complication".

The paper is organized as follows. In Section 1, we introduce the basic assumptions and notation. In Section 2, we present a general characterization of the set of equivalent martingale measures in the presence of default risk, and we introduce a formal definition of the market price for default risk. In Section 3, we derive an explicit expression for the market price of default risk using a defaultable extension of a standard continuous-time consumption-based CAPM. In Section 4, we discuss an application to the valuation of employee compensation packages. A conclusion and suggestions for further research can be found in Section 5, while proofs of some results and technical details are relegated to an appendix.

1 Assumptions and Notation

Uncertainty in the economy is described through a probability space $(\;\;, \mathcal{A}, \mathbb{P})$. Of particular interest is some risky asset (that we interpret as the market portfolio in Section 3), the price

[4]Still from Rubinstein (1995): "(...) the probability of forfeiture is no doubt negatively correlated with the success of the corporation. In particular, if the underlying stock price rises over the life of the options and perforce the options become quite valuable, employees are probably less likely to be fired or leave their jobs voluntarily". If one further assumes that the company's stock has a positive beta, one concludes that there is a negative correlation between the return on the market portfolio and the instantaneous probability of forfeiture.

of which, denoted by S, is assumed to be given by[5]

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t \tag{1}$$

where $(W_t)_{t\geq 0}$ is a standard Brownian motion. We further assume that a risk-free asset is also traded in the economy. The return on that asset, typically a default free bond, is given by $\frac{dB_t}{B_t} = rdt$, where r is the risk-free rate in the economy[6]. The agents' basic information set is captured by a filtration $\mathbb{F} = \{\mathcal{F}_t; t \geq 0\}$, with $\mathbb{F} \in \mathcal{A}$, which is the augmented filtration generated by the standard Brownian motion W. Note that the market defined in equation (1) is complete because there is one source of randomness, the standard Brownian motion W, and one traded asset. Starting from a complete market situation will allow us to more easily focus on the specific form of incompleteness induced by uncertainty over date of default.

One security in the economy, which is the focus of our attention, is defined by a *promised* unit dividend supposed to be paid at time T, subject to the risk of potential default that may occur before T. If default occurs at some random time τ prior to T, then the asset pays either nothing, or more generally some fractional recovery amount paid at time τ. Under the assumption of no recovery upon default, the *actual* dividend is $\mathbf{1}_{\{\tau>T\}}$. Essentially, one may think of that security as a defaultable zero-coupon unit bond. The interpretation is standard: everything is as if the agent lives in a Lucas-tree economy, and consumes the aggregate production. The additional feature is that one of the trees that he agent is contemplating can stop producing consumption goods at a random time τ.

1.1 A Model of the Time of Default

Following a reduced-form approach to default risk, we do not attempt here to fully specify the mechanism that leads to default. The reason why we choose to do so is that we are primarily interested in situations such that the presence of an uncertain time-horizon induces some new uncertainty in the economy[7]. Therefore, in what follows, we shall instead simply consider that the date τ of default is a positive random variable measurable with respect to the sigma algebra $\mathcal{A}$ that *is not a stopping time of the filtration* $\mathbb{F}$ generated by asset prices. In other words, we do not assume that observing asset prices up to date t implies full knowledge about whether τ has occurred or not by time t. Formally, it means that there are some dates $t \geq 0$ such that the event $\{t < \tau\}$ is not $\mathcal{F}_t$-measurable. This constrasts to the structural approach

[5] More involved processes may be considered at the cost of added complexity. An extension to the multi-dimensional case, on the other hand, is straightforward.

[6] One may introduce stochastic interest rates at the cost of added complexity.

[7] This does not imply that the random time of default may not be dependent upon asset prices. On the contrary, it is, in general, dependent upon asset prices (see Section 3); yet it is not fully explained by past asset prices as in the case of a stopping time of the asset price filtration.

to corporate bond pricing. In particular, in Black and Cox (1976) or Longstaff and Schwartz (1995), default occurs at the first hitting time of a given barrier by a process describing the value of the assets of the firm. In that case, no new uncertainty is added to the economy by the presence of default, markets actually remain complete, and arbitrage valuation of risky debt can be performed.

On the other hand, we assume that the uncertain time of default τ admits a representation in terms of an *intensity* process (see El Karoui and Martellini (2000) for necessary and sufficient conditions). The intensity process for τ is a process λ that can be interpreted as the conditional rate of arrival of default at time $t \leq \tau$, given no default up to that time. In the life-insurance literature, it is usually known as a *force of mortality rate*, or *hazard rate* in reliability theory, denoting the fact that, for a small interval of time δt, the conditional probability at time t that an accident occurs between t and $t + \delta t$, given survival up to t, is approximately $\lambda_t \delta t$. Formally, we have[8], for $t < \tau$

$$\lambda_t = \lim_{\delta t \to 0} \frac{1}{\delta t} \frac{\mathbb{P}\left(t < \tau < t + \delta t \middle| \mathcal{F}_\infty\right)}{\mathbb{P}\left(t < \tau \middle| \mathcal{F}_\infty\right)} \tag{2}$$

This setup is similar to the one used in reduced-form models of default date (see for example Duffie and Singleton (1999) or Lando (1997), or Brémaud (1981) for a mathematical treatment). It should be noted that if λ is deterministic, then τ is independent of $\mathbb{F}$. When the intensity is a constant λ, τ is simply the date of the first jump of a standard Poisson process.

In order to describe the information structure of the economy in the presence of unpredictable default risk, we introduce a filtration, potentially larger than $\mathbb{F}$, which also encompasses the information about the realization of τ. The intuitive concept of "enlarging" an information set has been given formal content by Jeulin (1980). The objective is to define a filtration, denoted by $\mathbb{G} = \{\mathcal{G}_t; t \geq 0\}$ and known as the "progressive enlargement" of $\mathbb{F}$, as the smallest filtration containing $\mathbb{F}$ of which τ is a stopping time. To that end, we introduce[9] $\mathcal{N}_t = \sigma(\tau \wedge t)$, the filtration generated by the family $\tau \wedge t$, where $\tau \wedge t$ denotes $\inf(\tau, t)$. The filtration $\mathbb{G}$ is taken to be the smallest right continuous family of sigma-fields such that both $\mathcal{F}_t$ and $\mathcal{N}_t$ are in $\mathcal{G}_t$. In what follows, we shall maintain the assumption that $\mathbb{G}$ is the information set available to the agents in the economy. In other words, we assume that the information available to the agents at any date t ($\mathcal{G}_t$) encompasses information about past values of asset prices ($\mathcal{F}_t$), and also information about whether default has occurred or not ($\mathcal{N}_t$)[10].

[8] In what follows $\mathbb{P}\left(\tau > t \middle| \mathcal{F}_\infty\right)$ may be replaced by $\mathbb{P}\left(\tau > t \middle| \mathcal{F}_t\right)$ under assumption (3).

[9] The family $\mathbb{N} = \{\mathcal{N}_t; t \geq 0\}$ is not a right-continuous filtration and the so-called "standard conditions" (see for example Karatzas and Shreve (1991)) are not satisfied. For that reason, one technically needs to consider instead the *right regularization* of $\mathcal{N}$, $\mathbb{M} = \{\mathcal{M}_t; t \geq 0\}$, where $\mathcal{M}_t = \cap_{\epsilon > 0} \mathcal{N}_{t+\varepsilon}$.

[10] If τ is already a $\mathbb{F}-$stopping time, then no augmentation is needed and $\mathbb{G} = \mathbb{F}$. In this paper, we rather assume that τ is simply a measurable random variable, so that the enlargement described here is not a trivial operation.

1.2 A Technical Assumption

There are two sources of uncertainty related to asset pricing of defaultable securities, one stemming from the randomness of prices (market risk), the other stemming from the randomness of the timing of default τ (default risk)[11]. A serious complication is that, in general, these two sources of uncertainty are not independent. Separating out these two sources of uncertainty is a useful operation that may be achieved as follows. Conditioning upon $\mathcal{N}_\infty$ (i.e., upon τ) allows one to isolate a pure asset price uncertainty component: given a specific realization of τ, the only remaining source of randomness comes from asset prices. On the other hand, conditioning upon $\mathcal{F}_\infty$ allows one to isolate a pure default timing uncertainty component. Since $\mathcal{F}_\infty$ contains information about the whole path of risky asset prices, $\mathbb{P}[\tau > t|\, \mathcal{F}_\infty]$, for example, is the probability that default is still to happen at date t given all possible information about asset prices.

Some assumption is needed at this point to specify the exact nature of the relationship between asset price uncertainty and default uncertainty. An extreme assumption consists in taking $\mathbb{P}[\tau > t|\, \mathcal{F}_\infty] = \mathbb{P}[\tau > t]$ for all t. This is an independence assumption, which expresses that the timing of default is totally unrelated to asset prices. Such an assumption is a clear oversimplification. A natural, and more general, assumption in this context is

$$\mathbb{P}[\tau > t|\, \mathcal{F}_\infty] = \mathbb{P}[\tau > t|\, \mathcal{F}_t] \tag{3}$$

This condition, known as the *K-assumption* in probability theory (see for example Mazziotto and Szpirglas (1979)), will be maintained throughout the paper. Note that assumption (3) is implied by independence, since we have in this case $\mathbb{P}[\tau > t|\, \mathcal{F}_\infty] = \mathbb{P}[\tau > t|\, \mathcal{F}_t] = \mathbb{P}[\tau > t]$. Despite its technical character, condition (3) is a very natural assumption, the interpretation of which is as follows. It requires that the probability of default on a given contract (typically a zero-coupon bond issued by a firm) happening or not *before* time t does not depend upon knowledge about the whole market return process (captured by $\mathcal{F}_\infty$), including what happens *after* time t, but solely upon knowledge about asset returns *up to* time t (captured by $\mathcal{F}_t$). The important feature is that past, and not future, market returns may affect uncertainty about the timing of default. Under that formulation, the K-assumption appears as a desired feature in most reasonable financial context, ruling out at most features such as inside information. The loss of generality implied, however, is not as limited as the above discussion might suggest. Indeed, by defining risky asset returns as in equation (1), we implicitly do *not* allow the expected return and volatility of the risky asset to be affected by the event of default. In other words, roughly speaking, we are allowing default on a given firm to be affected by (past)

[11] There is a third source of uncertainty in the presence of uncertain recovery (see Duffie and Singleton (1999)).

market returns, but we are not allowing market returns to be affected by default on a given firm.

Assumption (3) is actually necessary because it is the most natural and general assumption about asset price uncertainty and timing uncertainty which is allowed to maintain tractability. More specifically, the following proposition holds.

Proposition 1 *Assumption (3) is a sufficient and necessary condition for all bounded $\mathbb{F}$-martingales to be also bounded $\mathbb{G}$-martingales.*

Proof. See El Karoui and Martellini (2000) or Elliott, Jeanblanc and Yor (2000)[12]. ■

Assumption (3) ensures that the introduction of an uncertain time-horizon and the use of an enlarged filtration do not dramatically affect various martingale-related properties used in asset pricing theory. In particular, it guarantees that the $\mathbb{F}$-Brownian motion W driving risky asset returns is also a $\mathbb{G}$-Brownian motion. Because that is not granted in general, assumption (3) has to be maintained in the literature on the reduced-form approach to default risk (e.g., Duffie and Singleton (1999) or Lando (1997)), even though this is generally not explicitly stated.

2 An Arbitrage Characterization of the Market Price for Default Risk

In what follows, we maintain the assumption $\mathbb{P}[t < \tau | \mathcal{F}_t] = \mathbb{P}[t < \tau | \mathcal{F}_\infty]$ for all t, and restrict[13] our search of EMMs to the set $\mathcal{E}$ of all probability measures $\mathbb{Q}$ equivalent to $\mathbb{P}$ that also satisfy that assumption, that is such that $\mathbb{Q}[t < \tau | \mathcal{F}_t] = \mathbb{Q}[t < \tau | \mathcal{F}_\infty]$ for all t. The following proposition provides an explicit characterization of any such equivalent martingale measure, as well as the relationship between the intensity process under the original and under a new equivalent measure.

Proposition 2 *The set $\mathcal{E}$ of all possible EMMs which satisfy the assumption (3) is given by*

$$\mathcal{E} = \left\{ \mathbb{Q}_{H,\beta};\, \exists\ H \text{ and } \beta,\, \mathbb{F}-\text{adapted processes, s.t. } \left.\frac{d\mathbb{Q}_{H,\beta}}{d\mathbb{P}}\right|_{\mathcal{G}_t} = \xi_1(t) \times \xi_2(t) \right\} \tag{4}$$

[12] A condition similar to the K-assumption also appears in Madan and Unal (1995) (equation B.9).

[13] See Section 3 for a discussion of that point in a CCAPM type of model.

with

$$\xi_1(t) = \exp\left(H_\tau \mathbf{1}_{\{\tau \leq t\}} - \int_0^{t\wedge\tau} \left(e^{H_s} - 1\right) \lambda_s ds \right) \tag{5}$$

$$\xi_2(t) = \exp\left(-\int_0^t \beta_s dW_s - \frac{1}{2}\int_0^t \beta_s^2 ds \right) \tag{6}$$

Furthermore, for all β_t, τ admits the intensity process

$$\widehat{\lambda}_t = e^{H_t}\lambda_t \tag{7}$$

under $\mathbb{Q}_{H,\beta}$, where λ_t is the intensity process under the original measure $\mathbb{P}$.

Proof. See Appendix 1. Note that this is subject to the usual integrability conditions[14] $\mathbb{E}^{\mathbb{P}}(\xi_1) = 1$ and $\mathbb{E}^{\mathbb{P}}(\xi_2) = 1$. Changes of measure for discontinuous processes have been used for example in insurance literature by Aase (1999), Delbaen and Haezendonck (1989) and Sondermann (1991), and by Jarrow and Madan (1995) and Jarrow and Turnbull (1995) in finance literature. A classic mathematical reference is Brémaud (1981). ∎

A probability measure $\mathbb{Q}$ in $\mathcal{E}$, equivalent to $\mathbb{P}$, shall be regarded as a risk-neutral measure with respect to both asset price and default risks. Note the convenient multiplicative separation of asset price and default risk-adjustments, a result essentially driven by assumption (3). That separation result, however, is somewhat deceiving; in particular it does *not* imply that ξ_1 is a pure credit risk adjustment. On the contrary, one may in general expect ξ_1 to contain some market risk component, because we know that part of credit risk is driven by market risk. One may even argue that in a CAPM world, any pure credit risk component, that is any component independent of market risk, shall not be priced in equilibrium. These questions are discussed in some detail in Section 3.

The term ξ_2 solely affects the asset return process defined in equation (1) and has no impact on the intensity process of time of default. It provides a pure market risk adjustment; it is the standard adjustment to the original probability of various price path scenarios performed by investors to account for aversion with respect to asset price risk. Hence, β is the traditional *market price of market risk.* Within the context of our model, because there is one random perturbation and one traded asset, it is uniquely defined as $\beta = \frac{\mu - r}{\sigma}$. On the other hand, the term ξ_1 is the Radon-Nikodym derivative of a change of measure with respect to uncertainty in timing of default. It affects the intensity process of the uncertain time but not the return process, and captures a market adjustment for default risk.

We thus obtain the following definition for the market price for default risk.

[14] A sufficient condition for $\mathbb{E}^{\mathbb{P}}(\xi_2) = 1$ is the Novikov condition (see Karatzas and Shreve (1998)). A sufficient condition for $\mathbb{E}^{\mathbb{P}}(\xi_1) = 1$ is given by theorem 11, Section VI in Brémaud (1981).

Definition 1 *The market price for default risk is defined as the logarithm of the ratio of the risk-adjusted intensity of default to the original intensity of default, that is* $H_t = \ln \frac{\widehat{\lambda}_t}{\lambda_t}$.

Hence a zero market price for default risk coincides with no adjustment for time-horizon probability distribution. Using the original intensity process λ for pricing purposes is equivalent to making the assumption of risk-neutrality with respect to timing risk. In this case, $\xi_t = \xi_2(t)$. A possible justification would be that timing risk may be diversified away (see Section 3 for a sufficient condition[15]); in all other cases, a risk-neutral intensity process $\widehat{\lambda}$ should be used for pricing purposes[16]. In Section 3, we provide a derivation of the market price for timing risk in a standard equilibrium setup.

We now recall a general pricing formula for a defaultable security. This result is central in this literature and appears in particular in Duffie, Schroder and Skiadas (1996), Duffie and Singleton (1999), Duffie and Lando (1997), Jarrow and Turnbull (1992, 1995), Madan and Unal (1994), or Lando (1997). Because, the result is not new, we do not report the proof here (see for example Duffie and Singleton (1999) or Lando (1997)). It is an interesting result, since it states that standard term structure modeling techniques apply for defaultable bonds, provided that one uses a generalized short-term interest rate given by $R_t = r_t + \widehat{\lambda}_t$.

Proposition 3 *The price of a* $\mathcal{F}_T-$*measurable promised cash-flow* X_T *received at date* T *unless default occurs before date* T*, is given by*

$$\begin{aligned} p_t^d &\equiv \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T r_s ds\right) X_T \mathbf{1}_{\{\tau > T\}} \middle| \mathcal{G}_t\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T \left(r_s + \widehat{\lambda}_s\right) ds\right) X_T \middle| \mathcal{F}_t\right] \end{aligned} \tag{8}$$

The intuition is straightforward (see for example Duffie and Singleton (1999)). In a one-period setting, it states that the value of \$1 received at date Δt unless default occurs is the

[15] In a different setup, see also Jarrow, Lando and Yu (2000) for conditions under which the intensity process remains unchanged under the equivalent martingale measure. In general, because of the presence of some systematic risk, some adjustment to the actual probability of default is to be expected for pricing purposes (see Section 3).

[16] This is similar to option pricing in the presence of stochastic volatility. For example, in Hull and White (1987), stochastic volatility risk is assumed not to be rewarded. While a "price" is obtained under that assumption, no perfect hedging strategy is possible due to a market incompleteness induced by stochastic volatility. Another example is option pricing when the underlying asset follows a mixed diffusion-jump process. In Merton (1976), it is assumed that jump risk is non-systematic, and hence not rewarded in a CAPM framework. When jump risk can not be diversified away, however, one needs to derive the price for jump risk by some equilibrium argument (see for example Naik and Lee (1990), Ahn (1992) and Chang and Chang (1996)).

discounted value $e^{-r\Delta t}$ of \$1 multiplied by the *risk-adjusted* probability of getting it, that is $1-\widehat{\lambda}\Delta t$. Using the approximation $1-\widehat{\lambda}\Delta t \simeq e^{-\widehat{\lambda}\Delta t}$, we obtain $p_t^d = e^{-\left(r+\widehat{\lambda}\right)\Delta t}$, to be compared to (8).

3 A Continous-Time Defaultable Consumption-Based Capital Asset Pricing Model

While the absence of arbitrage opportunities implies that the set $\mathcal{E}$ is not empty (see Harrison and Kreps (1979) and Harrison and Pliska (1981)), uniqueness, on the other hand, is not granted. Even when the asset markets are complete in the sense that market risk is spanned by existing securities, as is the case here, and ξ_2 is uniquely defined through equation (6), there still exists an infinite number of possible equivalent martingale measures consistent with the absence of arbitrage. This is because uncertainty about timing induces a specific form of market incompleteness in the general case when the random time is not a stopping time of the asset filtration. In other words, unless there exists some traded security that spans the uncertainty related to default risk, no understanding of which EMM in $\mathcal{E}$ should be used can be obtained by only assuming the absence of arbitrage. Further justification for a specific choice of an EMM among all possible elements of $\mathcal{E}$ is needed; this amounts to specifying a choice for the risk-adjusted probability of default $\widehat{\lambda}$, or equivalently for the market price of timing risk H. Given that any positive value of $\widehat{\lambda}$ is consistent with the absence of arbitrage, this can only be obtained by using some equilibrium argument[17]. We now present a derivation of the market price for default risk in a continuous-time equilibrium model.

To answer that question, we cast the problem within the context of a pure exchange economy similar to the one in Lucas (1978). The economy is populated by m identical agents with infinite time-horizon. We use the assumption of homogenous agents because it is a convenient way to ensure the existence of a representative agent in the absence of complete markets[18]. We denote by u the utility function common to all agents, where u is a continuous, strictly increasing, strictly concave and continuously differentiable function defined on $(0,\infty)\times(0,\infty)\longrightarrow\mathbb{R}$. Uncertainty in the economy is described through a probability space $(\ \ ,\mathcal{A},\mathbb{P})$. We assume that aggregate endowment follows an Ito process

$$de_t = \mu_e dt + \sigma_e dW_t^e \tag{9}$$

[17] In a complete markets situation, an alternative solution for narrowing down to one the number of possible EMMs consists in using market prices of a redundant security to perform relative pricing.

[18] An alternative way of obtaining the existence of a representative agent would have precisely consisted in assuming complete markets. Because the presence of default risk may induce a specific form of incompleteness on which we focus in this paper (see Section 1), that is not the chosen approach.

where W^e is a standard Brownian motion defined on the probability space $(\ , \mathcal{A}, \mathbb{P})$.

One security in the economy, which is the focus of our attention, is defined by a *promised* unit dividend supposed to be paid at time T, subject to the risk of potential default that may occur before T. Hence, the *actual* dividend is $\mathbf{1}_{\{\tau>T\}}$(we assume here no recovery upon default for simplicity). Essentially, one may think of that security as a defaultable zero-coupon unit bond. We do not attempt here to fully specify the mechanism that leads to default. Our goal is rather to obtain simple and useful insights about the market price for default risk in a stylized model. The interpretation is standard: everything is as if the agent lives in a Lucas-tree economy, and consumes the aggregate production. The additional feature is that one of the trees that he agent is contemplating can stop producing consumption goods at a random time τ.

Here τ is an exogenous random time of default defined in terms of an intensity process λ, as explained in Section 1. For concreteness, let us specify a mean-reverting process[19] for the intensity of default. More specifically, we assume that

$$d\lambda_t = a\left(b - \lambda_t\right) dt + \sigma_\lambda dW_t^\lambda \tag{10}$$

where $\left(W^\lambda\right)_{t\geq 0}$ is a standard Brownian motion with $\mathbb{E}_t\left(dW_t^e dW_t^\lambda\right) = \rho dt$. This term captures the correlation between the instantaneous probability of default and aggregate endowment. Since the probability of default on a given firm is likely to increase as economic growth decreases, one may intuitively expect to have $\rho < 0$. Whether that correlation is significantly different from zero is ultimately a matter of empirical investigation, that may be tested by using data at the individual firm level, or preferably aggregate data on rating migrations. The specification in equations (9) and (10) is convenient because it allows us to capture a possible dependence of the event of default on the market return, without having to specify the exact mechanism of how default is actually triggered. For example, the fact that the probability of default increases as the return on the market decreases is consistent with an explicit model of default triggered by the value of the assets of the firm reaching a given boundary, as long as the projects undertaken by the firm are positive beta assets.

We assume that the agents' information set is given by $\mathbb{G} = \{\mathcal{G}_t; t \geq 0\}$ that is the progressive enlargement of $\mathbb{F}$, that we now interpret as the filtration generated by the two-dimensional Brownian motion $\left(W^e, W^\lambda\right)$. We also maintain the assumption $\mathbb{P}[\tau > t|\, \mathcal{F}_\infty] = \mathbb{P}[\tau > t|\, \mathcal{F}_t]$. It is important to note that we do not allow the aggregate endowment process to be impacted by the event of default. This can be seen from the fact that the aggregate endowment process

[19] Note that a mean-reverting process allows the intensity term to take on negative values with positive probabilities, a problem which also applies to the Vasicek (1977) term structure model. The rule of thumb is that the problem may be neglected as long as the probability of getting negative values stays sufficiently small for the relevant values of the parameters.

is actually modelled as a continuous process with no (negative) jump at date τ (equation (9)). One key motivation for not including a jump component at date of default in the aggregate endowment is that this would imply a violation of the K-assumption. Indeed, conditioning upon the fact that a jump in the aggregate endowment occurs at some date $\tau = t_1$, agents would know that default has occurred at the same date t_1. In other words, we would have then $\mathbb{P}[\tau > t | \mathcal{F}_\infty] \neq \mathbb{P}[\tau > t | \mathcal{F}_t]$ because knowledge about the aggregate endowment up to infinity (given by $\mathcal{F}_\infty$, or rather $\mathcal{F}_T$ because time-horizon is assumed to be finite here) would provide information about the timing of default, because default would coincide with the date when aggregate endowment jumps. For example $\mathbb{P}[\tau > t | \mathcal{F}_\infty]$ would be equal to 1 if $t_1 > t$, while $\mathbb{P}[\tau > t | \mathcal{F}_t] = \exp\left[-\int_0^t \lambda_s ds\right] \neq 1$. On the other hand, the specification in equations (9) and (10) is clearly consistent with the K-assumption (under the original measure $\mathbb{P}$)[20]. A justification for having a continuous process for the aggregate endowment, while one security in the economy is subject to default risk, would be that we consider an event that is not significant enough to affect the global wealth in the economy. In other words, we assume that the dividend of the defaultable bond is very small compared to aggregate endowment. Note that this is not equivalent to stating that default risk is diversifiable. Actually, because one may expect some correlation between the aggregate endowment and the probability of default, default risk is not independent of market risk, even though the promised payoff does not represent a significant fraction of the aggregate endowment. We actually show that default risk is not, in general, diversifiable, and therefore should be priced (see equation (12))[21].

Asset pricing can be done in a convenient way by specifying the joint dynamics of asset payoffs and the *pricing kernel*[22], the value of which at date t is denoted by π_t. The covariance with the pricing kernel can be interpreted as systematic risk, corresponding, for example, with "beta" under the mean-variance CAPM. It is well-known that, under suitable assumptions (see Duffie and Zame (1989)), one may express the dynamics of the pricing kernel in terms of the marginal utility of a representative[23] agent's consumption $\pi_t = u_c(e_t, t)$, where u_c indicates a derivative in the customary way. In particular, one may write the dynamics of the state price deflator

$$\frac{d\pi_t}{\pi_t} = -rdt - \beta dW_t^e$$

[20] Maintaining the K-assumption under the original probability measure guarantees that the K-assumption is also satisfied under the equivalent martingale measure, at least within an additive time-separable utility framework.

[21] We refer the reader to Jarrow and Wu (2000) for an interesting model in which default on a small number of firms has an economy-wide impact because of the presence of significant counterparty risk.

[22] It is also known as a *state-price deflator* or a *stochastic discount factor*.

[23] Again, the existence of a representative agent in the absence of complete markets is ensured here by the fact that agents are identical, so that aggregation becomes a trivial operation.

where r and β may be interpreted as the risk-free rate and the market price of risk, respectively. We assume a constant risk-free rate because we want to focus in this paper on default risk, as opposed to interest rate risk.That assumption can be relaxed at the cost of added complexity.

Specifying the joint dynamics of asset payoffs and the state-price deflator provides a convenient way of pricing assets paying off general dividends. Hence, the price at time t of an asset that pays X_T consumption units at time T is $\mathbb{E}_t\left[\frac{\pi_T}{\pi_t}X_T\right]$. We now attempt to obtain insights about the equilibrium price of the defaultable unit zero-coupon bond with payoff $\mathbf{1}_{\{\tau>T\}}$. From the basic pricing equation, we obtain that p_t^d, the price at date t of a defaultable payoff $\mathbf{1}_{\{\tau>T\}}$, is

$$p_t^d = \mathbb{E}\left[\left.\frac{\pi_T}{\pi_t}\mathbf{1}_{\{\tau>T\}}\right|\mathcal{G}_t\right]$$

Using equation (8) from Section 2, we transform this expression into

$$p_t^d = \mathbb{E}\left[\left.\exp\left(-\int_t^T \lambda_s ds\right)\frac{\pi_T}{\pi_t}\right|\mathcal{F}_t\right]$$

where the expectation is taken under the original measure. From equation (7), that quantity must also coincide with the following expression under the risk-adjusted measure

$$p_t^d = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T \widehat{\lambda}_s ds\right)\right] = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T e^H\lambda_s ds\right)\right]$$

The following proposition provides a closed-form expression for the price of the unit defaultable zero-coupon bond and an explicit expression for the market price of default risk. (More general expressions for a random payoff X_T can easily be obtained using the same technique.)

Proposition 4 *We denote by* $p_t^d = \mathbb{E}\left[\left.\frac{\pi_T}{\pi_t}\mathbf{1}_{\{\tau>T\}}\right|\mathcal{G}_t\right]$ *the price at date t of a defaultable zero-coupon bond, with maturity date T and unit face value, subject to default at date τ with an intensity process defined in (10). The price at date* 0 *of that defaultable unit zero-coupon bond is given by*

$$p_0^d = \exp\left(-rT - \left(\lambda_0 T + \frac{\sigma_\lambda^2}{2a}T^2 - 2\frac{\sigma_\lambda\beta\rho}{a}T\right)\right) \tag{11}$$

Furthermore, if we assume that the market price for default risk is a constant H, then it is given by

$$H = \gamma\sigma_\lambda\beta\rho \tag{12}$$

where $\gamma = -\frac{2}{a\lambda_0+\sigma_\lambda^2 T}$ *is a negative number.*

Proof. See Appendix 2. ■

Remember that β is the market price of market risk. In other words, the market price of default risk is a linear function of the market price of market risk, with a slope coefficient proportional to $\rho\sigma_\lambda$. To interpret the expression (12), let us first consider a case with deterministic intensity. In that case, we have $\sigma_\lambda = 0$ and therefore $H = 0$. The same result is obtained in a case with stochastic intensity uncorrelated to aggregate endowment. In this case, we have $\rho = 0$ which also implies a trivial market price for default risk $H = 0$. Hence, agents price defaultable securities by discounting their promised payoff with an adjusted interest rate equal to the risk-free rate augmented by the original instantaneous probability of default. In other words, $\widehat{\lambda} = \lambda$. The intuition is that when the probability of default is not correlated to aggregate consumption, default risk is non-systematic and not rewarded. Hence, one may use the original probability of default λ in the pricing formulae. When the intensity of default is correlated to aggregate endowment growth, then agents use a risk-adjusted probability of default when pricing defaultable securities different from the original probability of default. This is consistent with economic intuition. The term σ_λ is a measure of the intensity of intensity risk, and the term ρ is a measure of the correlation between default risk and market risk.

It is well-documented that peaks in likelihood of default coincide with periods of economic distress (see for example Blume, Keim and Sandeep (1991)). Consistent with the intuition that the probability of default on a given firm increases as the economy is slowing down, let us assume that $\rho < 0$. This translates into a positive market price for default risk, since $\gamma\sigma_\lambda\beta\rho$ is a positive number when ρ is negative (recall that γ is always negative). In other words, the risk-adjusted probabilities of default are higher than the empirical probabilities of default. The intuition is as follows. Default risk is priced in equilibrium because it is not diversifiable: default hurts more because it occurs more frequently in those states of the world where the aggregate endowment tends to be low. Our finding that the market price for market risk is a key ingredient of the market price for default risk is hardly surprising. For example, Elton et al. (2000) report that "while state taxes explain a substantial portion of the difference (between corporate and treasury rates), the remaining portion of the spread is closely related to the factors that we commonly accepted as explaining risk premiums for common stocks". In that context, expression (12) is convenient because it provides an explicit quantitative estimate of the magnitude of the effect.

It should be noted that our model predicts stylized facts that are consistent with salient empirical findings. First, note from equation (11) that the spread over the risk-free rate for maturity T is given by $\lambda_0 + \frac{\sigma_\lambda^2}{2a}T - 2\frac{\sigma_\lambda\beta\rho}{a}$. From that expression, we obtain the following results. First, our model predicts that the term structure of credit spreads should be upward sloping. This is consistent with the empirical findings of Helwege and Turner in the context of speculative-grade debt (1999). Traditional structural models of default risk can only be made compliant with these salient features by accounting for imprecisely-measured firm value (Duffie and Lando (2000))[24]. Also, our model predicts a non trivial spread $\lambda_0 - 2\frac{\sigma_\lambda\beta\rho}{a}$ (which

[24] One may also introduce jumps in the firm-value process or a liquidity premium in the model to make the

is always a positive number when ρ is positive) in the limit of very short maturities. While this a salient empirical feature of the term structure of credit spreads, it is not captured in most structural models of default (see Collin-Dufresne and Goldstein (2001) for an interesting attempt to account for that fact in a model with mean-reverting leverage ratios).

4 An Application to the Valuation of Employee Compensation Packages

We now turn to a one agent economy and provide some insights into the private value[25] (shadow price) set by a manager to compensation packages subject to vesting risk. Since, we focus on the question of vesting risk, we solely consider here for simplicity the case of a cash compensation contract. This allows us to abstract away from other layers of complexities involved in the valuation of stock options for example.

More specifically, we assume that, at date 0, an employee is granted a cash bonus contract that will be vested at date T, if and only if the employee is still with the company at that date, that is if and only if $\tau > T$, where τ is the random date of the employee's (voluntary or involuntary) departure from the company. From an asset pricing standpoint, this is equivalent to valuing an asset subject to default risk; at date T the employee shall receive a certain[26] cash-flow X, provided that $\tau > T$.

If $\tau < T$, we assume that she will receive nothing (similar to default with no recovery). From equation (8), the value at date 0 of that contract, denoted by C_0, is

$$C_0 = Xe^{-\left(r+\widehat{\lambda}\right)T} \tag{13}$$

That expression makes intuitive sense. It is the discounted (risk-neutral) average payoff of the contract. It has a present value Xe^{-rT} if the employee does not leave the company, an event which occurs with risk-adjusted probability $e^{-\widehat{\lambda}T}$, and zero, if the employee has left the company, an event which happens with probability $1-e^{-\widehat{\lambda}T}$. Note that C_0 goes to Xe^{-rT} as $\widehat{\lambda}$ goes to zero, as it should: if the probability of the employee leaving the company is zero, then she is certain to receive the cash bonus with discounted value Xe^{-rT}. On the other hand, C_0 goes to 0 as $\widehat{\lambda}$ goes to infinity: if the employee is about to leave the company with probability 1, then she is certain not to receive the promised payoff, so that contract has a zero value. As an illustration, consider the value of a $X = \$100$ compensation package with a time before

predictions of standard structural models consistent with a strictly positive spread for infinitesimal maturities.

[25] Hence we consider the value of the compensation package from an employee's, as opposed to the company's, standpoint (see Martellini and Urošević (2000) for an analysis of the difference between the two).

[26] The case of a random payoff (e.g., stock or stock option compensation packages) may be addressed at the cost of additional complexity.

vesting $T = 1$ year. For a risk-adjusted intensity $\widehat{\lambda} = .7$, which corresponds to a risk-adjusted average time $1/\widehat{\lambda}$ before leaving the company equal to 1.4 years, the value of the compensation package drops below \$50, i.e. it is worth less than 50% of what it would be worth if the stock had been received without any vesting restriction

We now use the analysis developed in Section 3 to obtain a better understanding of what the risk-adjusted intensity should be. For simplicity, we discuss below the case of a compensation package with a constant payoff, for example a cash bonus to be received at date T if the employee has not left the company before that date. From equation (12), we get $\widehat{\lambda} \simeq \lambda e^{\gamma \sigma_\lambda \beta \rho}$. Figure 1 displays the value of the compensation package, which we again denote by SC, as a function of the volatility of the intensity process σ and the correlation coefficient ρ between the intensity process of departure from the company and the endowment process.

In Figure 2, we set the volatility of the intensity process to the value $\sigma_\lambda = .2$ and consider the value of the compensation package as a function of the correlation ρ between the endowment process and the intensity process. Given that the likelihood of a given employee leaving her company increases as the stock price of this company decreases, we expect the correlation coefficient ρ to be negative for positive beta stocks. This tends to lower the value of the compensation package. The intuition again is that vesting risk is not diversifiable because the states of the world in which the employee leaves the company correspond to states of the world in which her wealth is relatively low (this, again, holds for the case of a positive beta stock). To get a better understanding of the magnitude of the effect, let us note that the value of the compensation package ranges from \$65 to \$85 as the correlation coefficient increases from -1 to $+1$.

Conversely, a positive correlation tends to increase the value of the compensation package. Let us for example discuss the case of a Silicon Valley high level engineer who may at any point leave for another company. In that case, she expects a positive shock to her endowment at date τ, because of some big bonus which shall more than compensates for the loss of the promised package (otherwise she would have no financial incentive to leave her previous employer). One may describe such a situation by allowing the correlation ρ between the endowment process and the intensity process to be positive[27], which implies finally that $\widehat{\lambda} < \lambda$. Hence, one concludes that the risk-adjusted probability of not obtaining the compensation package is lower than the original probability, which in turn implies a higher value for the compensation package. This is not because the employee likes risk; it is because she is not so much hurt by non-vesting, since non-vesting means leaving the company, an event which tends to occur when her endowment process is affected by a positive shock.

[27] Arguably such a situation could be better describe by introducing a jump component in the agent's endowment process.

5 Conclusion

This paper provides an attempt to address the issue of the market price of default risk from a theoretical perspective, and may be regarded as filling in a gap in the literature about credit risk by providing a first step in the direction of an equilibrium framework for the pricing of defaultable securities in an incomplete market setup. Our research may be extended in several directions, some of which are currently being developed by the authors. First, it would be desirable to provide an inspection of the equilibrium price for default risk under more general conditions. In particular, at the cost of added complexity, the model could be extended to a setup with non trivial recovery and stochastic interest rates. Given that the model leads to testable implications, another potentially interesting question is to perform an empirical testing of whether default timing risk is priced in equilibrium, and what is the magnitude of the effect. This may help identify a pure credit component in the spread of defaultable bonds over default-free securities. Also, one may develop specific applications of the theory to a variety of potential applications, including for example the valuation of credit derivatives, but also CAT bonds[28] or mortgage-backed securities. This may be done in a tractable framework with a stochastic intensity process modeled as a Markov process. Finally a related question of practical and theoretical interest is optimal consumption and investment in the presence of default risk.

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A Appendix

A.1 Equivalent Martingale Measures and Default Risk

Let us denote by $\widehat{\lambda}_s = e^{H_s}\lambda_s$ the intensity process of τ under $\mathbb{Q}$, for some process H_s. Then, we have $\mathbb{Q}[t < \tau | \mathcal{F}_\infty] = \exp\left(-\int_0^t \widehat{\lambda}(s)\,ds\right)$. From this, it follows that there is some process ξ_2 such that

$$\exp\left(-\int_0^t \widehat{\lambda}(s)\,ds\right) = \mathbb{E}^{\mathbb{P}}[\xi_2(\tau)\,\mathbf{1}_{\{\tau\leq t\}} | \mathcal{F}_\infty] = \int_t^\infty \xi_2(s)\,\lambda(s)\,e^{-\int_0^s \lambda(u)du}ds$$

By integration by part, we have

$$\exp\left(-\int_0^t \widehat{\lambda}_s ds\right) = \int_t^\infty \widehat{\lambda}(s)\,e^{-\int_0^s \widehat{\lambda}(u)du}ds$$

and we obtain

$$\widehat{\lambda}(s)\,e^{-\int_0^s \widehat{\lambda}(u)du} = \xi_1(s)\,\lambda(s)\,e^{-\int_0^s \lambda(u)du}$$

or, using $\widehat{\lambda}_s = e^{H_s}\lambda_s$

$$e^{H_s}\lambda(s)\,e^{-\int_0^s \lambda(u)e^{H_u}du} = \xi_2(s)\,\lambda(s)\,e^{-\int_0^s \lambda(u)du}$$

from which we finally get

$$\xi_2(s) = \exp\left(H_s - \int_0^s \left(e^{H_s} - 1\right)\lambda_u du\right)$$

To conclude the proof, one needs to show that a separation holds between asset price and timing risk adjustments. By a monotone class argument, it is enough to show the result for any function of the form $H_t \mathbf{1}_{\{\tau>t\}}$ where H_t is a bounded function measurable with respect to $\mathcal{F}_t$. We have

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}[H_t \mathbf{1}_{\{\tau>t\}}] &= \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}[H_t \mathbf{1}_{\{\tau>t\}} \middle| \mathcal{F}_\infty]\right] = \mathbb{E}^{\mathbb{Q}}\left[H_t \mathbb{E}^{\mathbb{P}}[\mathbf{1}_{\{\tau>t\}} \xi_2(t) \middle| \mathcal{F}_t]\right] \\ &= \mathbb{E}^{\mathbb{P}}\left[\mathbf{1}_{\{\tau>t\}} \xi_2(t) \mathbb{E}^{\mathbb{P}}[H_t \xi_1(t) | \mathcal{F}_t]\right] = \mathbb{E}^{\mathbb{P}}[\xi_1(t)\, \xi_2(t)\, H_t \mathbf{1}_{\{\tau>t\}}]\end{aligned}$$

where we have used the assumption (3) in the second equality and the law of iterated expectations in the first and fourth equality. This concludes the proof.

A.2 Market Price of Default Risk

We need to compute

$$p_t^d = \mathbb{E}\left[\frac{\pi_T}{\pi_t} \mathbf{1}_{\{\tau>T\}} \middle| \mathcal{G}_t\right] = \mathbb{E}\left[\exp\left(-\int_t^T \lambda_s ds\right) \frac{\pi_T}{\pi_t} \middle| \mathcal{F}_t\right]$$

First note that for the process given in equation (10), $\int_t^T \lambda_s ds$ is Gaussian, so that we obtain

$$\mathbb{E}_t[\exp(x_t)\exp(y_t)] = \exp\left(\mathbb{E}_t(x_t) + \frac{1}{2}\mathbb{V}_t(x_t) + \mathbb{E}_t(y_t) + \frac{1}{2}\mathbb{V}_t(y_t) + \mathbb{C}ov_t(x_t, y_t)\right)$$

where we have defined $x_t \equiv -\beta\left(W_T^e - W_t^e\right) - \left(r + \frac{\beta^2}{2}\right)(T-t)$ and $y_t \equiv -\int_t^T \lambda_s ds$. We note that $\mathbb{E}_t(x_t) + \frac{1}{2}\mathbb{V}_t(x_t) = -r(T-t)$. We also introduce some notation

$$m_{t,T} = \mathbb{E}_t\left[\int_t^T \lambda_s ds\right]; \; v_{t,T} \equiv \mathbb{V}_t\left[\int_t^T \lambda_s ds\right]; \; c_{t,T} \equiv \mathbb{C}ov_t(x, y)$$

and rewrite the price of the defaultable bond as

$$p_t^d = \exp\left(-r(T-t) - m_{t,T} + \frac{1}{2}v_{t,T} + c_{t,T}\right)$$

Equation (10) can be integrated to give

$$\lambda_t = \lambda_0 \exp(-at) + b(1 - \exp(-at)) - \sigma_\lambda \int_0^t \exp(-a(t-s))\, dW_s^\lambda$$

and we have, using standard results (see for example Martellini and Priaulet (2000))

$$m_{t,T} = b\,(T-t) + (\lambda_t - b)\,\frac{e^{-at}}{a}\left(1 - e^{-a(T-t)}\right)$$

and

$$v_{t,T} = \frac{\sigma_\lambda^2}{2a^3}\left(1 - e^{-a(T-t)}\right)^2 + \frac{\sigma_\lambda^2}{a^2}\left((T-t) - \frac{1 - e^{-a(T-t)}}{a}\right)$$

One may also check that (where we take $t = 0$)

$$\begin{aligned}
\mathbb{C}ov\,(x,y) &= -\mathbb{C}ov\left(\sigma_\lambda \int_0^T ds \int_0^s \exp\left(-au\right) dW_u^\lambda, \beta W_T^e\right) \\
&= -\sigma_\lambda \beta \int_0^T ds \int_0^s \exp\left(-au\right) \mathbb{C}ov\left(dW_u^\lambda, W_T^e\right) \\
&= -\sigma_\lambda \beta \rho \int_0^T ds \int_0^s \exp\left(-au\right) du = \frac{\sigma_\lambda \beta \rho}{a} \int_0^T ds \left(1 - \exp\left(-as\right)\right) \\
&= -\frac{\sigma_\lambda \beta \rho}{a}\left(T - \left(\frac{\exp\left(-aT\right) - 1}{a}\right)\right)
\end{aligned}$$

Using the approximation $\exp(\varepsilon) \simeq 1 + \varepsilon$ for small ε, we finally obtain $m_{0,T} \simeq \lambda_0 T$, $v_{0,T} \simeq \frac{\sigma_\lambda^2}{2a} T^2$ and $c_{0,T} \simeq -2\frac{\sigma_\lambda \beta \rho}{a} T$. Finally we obtain

$$p_0^d = \exp\left(-rT - \left(\lambda_0 T + \frac{\sigma_\lambda^2}{2a} T^2 - 2\frac{\sigma_\lambda \beta \rho}{a} T\right)\right)$$

Then, we use the assumption of a constant market price for default risk H, and the following identification (see equation (8), where we note that $e^H \lambda_s = \widehat{\lambda}_s$)

$$p_0^d = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T e^H \lambda_s ds\right)\right] = \exp\left(-rT - \left(e^H \lambda_0 T + e^{2H}\frac{\sigma_\lambda^2}{2a} T^2\right)\right)$$

Using $e^x \simeq 1 + x$ for small x, we get

$$p_0^d \simeq \exp\left(-rT - \left(\left(1 + H\right)\lambda_0 T + \left(1 + 2H\right)\frac{\sigma_\lambda^2}{2a} T^2\right)\right)$$

Comparing to equation (11), we finally obtain

$$\left(1 + H\right)\lambda_0 T + \left(1 + 2H\right)\frac{\sigma_\lambda^2}{2a} T^2 = \lambda_0 T + \frac{\sigma_\lambda^2}{2a} T^2 - 2\frac{\sigma_\lambda \beta \rho}{a} T$$

which translates into

$$H = \frac{-2\frac{\sigma_\lambda \beta \rho}{a} T}{\lambda_0 T + \frac{\sigma_\lambda^2}{a} T^2} = \gamma \sigma_\lambda \beta \rho$$

where $\gamma = -\frac{2}{a\lambda_0 + \sigma_\lambda^2 T}$.

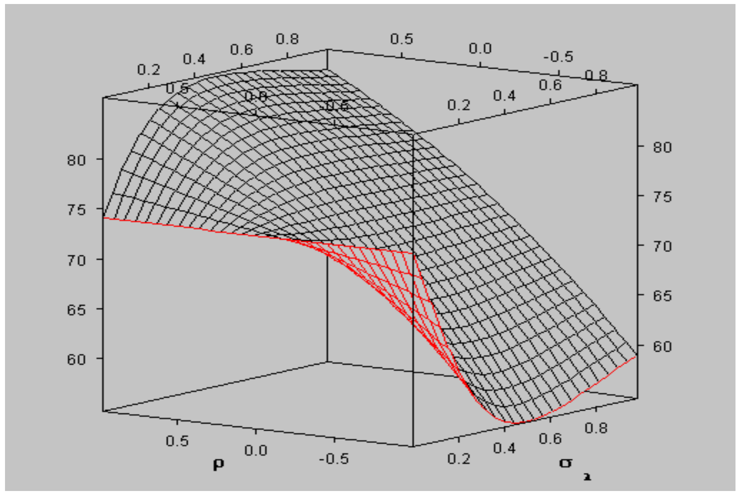

Figure 1: Value of the Compensation Package. This figure displays the value SC of a compensation package promising to pay \$100 in one year from now, with an initial intensity of departure under the true measure $\lambda = .25$ (which corresponds to a risk-adjusted average time $1/\lambda = 4$ years before leaving the company) as a function of the volatility of the intensity process σ and the correlation function ρ. We use a value for the risk-free rate $r = 5\%$, a speed of mean-reversion $a = 1$ and a risk-premium $\beta = \frac{\mu - r}{\sigma} = \frac{6\%}{.17} = .35$ (parameter values for the risk-premium are consistent with numbers from table 8.1, page 308, in Campbell, Lo, and MacKinlay (1997), obtained with annual data from 1889 to 1994).

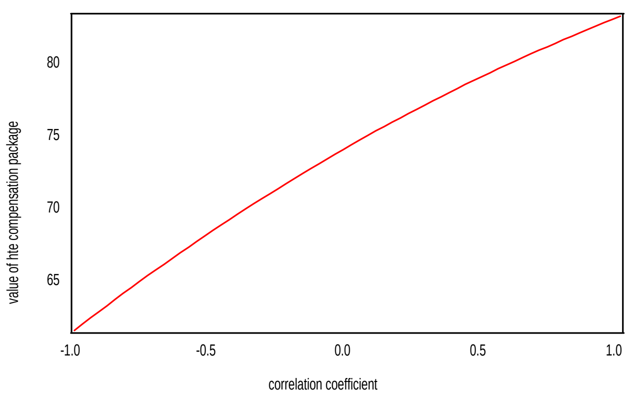

Figure 2: Compensation Package. This figure displays the value of a compensation package as a function of the correlation coefficient ρ for a value $\sigma_\lambda = .2$, under the same conditions as in figure 1. For comparison, note that the value of the same compensation package obtained under the original probability of nonvesting (or equivalently for $\rho = 0$) is equal to \$74.