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Effective and Empirical Durations of Mortgage Securities

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INTRODUCTION AND SUMMARY

Understanding the durations of mortgage-backed securities (MBSs) is critical for all participants in this huge market, yet the topic remains surrounded by misunderstanding and confusion. The object of this paper is to attempt to throw some light on this subject by theoretical and empirical analyses of both model- and market-based durations.

The first section of the paper reviews *effective durations*, which are derived from option-adjusted spread (OAS) models. While effective durations have become widely accepted, there is still some skepticism concerning their accuracy and usefulness. We describe the various assumptions that are part of standard effective duration calculations, and that typically lead to deviations between actual price moves and those predicted by effective durations. Appendix A derives a mathematical relationship for the actual price move in terms of the effective duration and the various other factors that have an impact on MBS prices, and this is used to obtain a general attribution formula for the discrepancy between actual and predicted price moves.

The second section discusses *empirical durations*, which are obtained by comparing actual MBS and Treasury price moves. Empirical durations are commonly used as sanity checks on model-based durations. But for them to be useful, it is necessary to understand the characteristics and biases of these statistical estimators. Appendix B derives the statistical properties of standard empirical duration estimators, and this in turn leads to a relationship between empirical and effective durations. This relationship can be used to intelligently combine the relevant information provided by empirical and effective durations, and leads to the concept of an *updated empirical duration.*

The third and final section compares the historical hedging performance of model- and market-based durations. Our conclusions are that if you have an OAS model that does *not* display a pronounced rate-dependence (i.e., the OASs are relatively flat across coupons, and correlations between OAS and yield changes are not persistant and stable), then the durations generated by the model will tend to outperform market-based durations over the long term.

Investors and traders use duration to estimate likely moves in the price of a security as rates change. For holders of MBSs, effective duration is a commonly used measure, given its ability to capture and incorporate variation in cash flow as rates change.

This section begins by explaining how to calculate effective duration, then compares projected and actual price changes, and finally discusses underlying assumptions in the calculation.

Calculation of Effective Duration

The typical steps in an effective duration calculation are as follows:

1. For a given price P, calculate the OAS.

2. Shift the yield curve upward in parallel by Δy bp and reprice the MBS at the original OAS. Call this price P^+.

3. Shift the yield curve downward in parallel by Δy bp and reprice the MBS at the original OAS. Call this price P^-.

4. Effective duration is then given by[1]

$$100 * \left(\frac{P^- - P^+}{P * 2 * \Delta y} \right) \tag{1}$$

Figure 1 illustrates this computation for FNMA 7.5s at the close of May 1, 1996.

Figure 1 . Effective Duration for a FNMA 7.5% Pass-Through

	3 Mo.	6 Mo.	1 Yr.	2 Yr.	3 Yr.	4 Yr.	5 Yr.	7 Yr.	10 Yr.	20 Yr.	30 Yr.
Treasury Curve	5.10	5.27	5.57	6.00	6.15	6.27	6.38	6.50	6.68	6.79	6.91

PRICE P = 98-25. Base Mortgage Rate = 7.64%

1. Calculate OAS. OAS = 65bp

2. Shift yield curve by +25bp. Reprice at 65bp OAS. P^+ = 97.542

3. Shift yield curve by -25bp. Reprice at 65bp OAS. P^- = 99.949

4. Calculate duration. Effective Duration = 100 * [(P^- - P^+) / (P * 0.50)] = 100 * [(99.949 - 97.542) / (98.781 * 0.50)] = 4.87

bp Basis points. OAS Option-adjusted spread. Treasury date: May 1, 1996.
Source: Salomon Brothers Inc.

Since $\Delta P / P \cong$ -(Effective Duration) * Δy, a duration of 4.87 indicates that if rates rally by 100bp, the price of the MBS is projected to increase by approximately 4.87%, assuming the various assumptions (constant OAS, parallel shifts, etc.) all hold. Similarly, if rates back up by 50bp, the price of the MBS is projected to decrease by 0.5*4.87%, or 2.44%. Note that since the effective

[1] Equation 1 is a numerical approximation to the exact formula (-1/P)*(dP/dy).

duration calculation uses 25bp yield-curve shifts, price movements for other shifts are obtained through linear extrapolation.

Comparing Projected Price Changes With Actual Price Changes

On May 1, 1996, the ten-year yield was 6.68%, and FNMA 7.5s had a price of 98-25 and an effective duration of 4.87. One month later, on May 31, 1996, the ten-year yield had increased to 6.85%. Figure 2 shows actual and projected price changes for the FNMA 7.5%.

Figure 2. Actual and Projected Price Changes, 1 May 96-31 May 96

Treasury Date	FNMA 7.5% Price	Ten-Year Yield	Eff. Dur.
May 1, 1996	98-25	6.675	4.87
May 31, 1996	97-30	6.850	
Actual Change:	-27 TICKS	+17.5bp	

Projected Price Change = (98.781) * (4.87%) * (-17.5%) = -0.840 = -27 TICKS

bp Basis points.
Source: Salomon Brothers Inc.

Although effective duration appears to have forecasted the price change for FNMA 7.5% very well in this example, can we safely conclude that this is always the case?

Figure 3 examines the actual and projected price changes for the period from June 7, 1996 to June 27, 1996. In this case, the actual price move was +6 ticks, while effective duration would have predicted a move of +12 ticks.

Figure 3. Actual and Projected Price Changes, 7 Jun 96-27 Jun 96

Treasury Date	FNMA 7.5% Price	Ten-Year Yield	Eff. Dur.
June 7, 1996	97-29	6.905	4.95
June 27, 1996	98-03	6.830	
Actual Change:	+6 TICKS	-7.5bp	

Projected Price Change = (97.906) * (4.95%) * (7.5%) = 0.363 = +12 Ticks

bp Basis points.
Source: Salomon Brothers Inc.

The relatively large disparity in this second example of -6 ticks between actual and projected price changes highlights the assumptions that are made when using effective duration to predict price movements.

Assumptions in Effective Duration Calculations

The main assumptions underlying the typical effective duration calculation include the following:

- Parallel yield-curve shifts

- Constant OAS pricing

- Constant term structure of volatility

- Constant current-coupon mortgage/Treasury spreads

- Absence of convexity• Absence of time value (carry)

Depending on which time period we focus on, some assumptions may hold, while others may be violated.

Parallel Yield-Curve Shifts

When using effective duration to predict price changes, the ten-year Treasury was treated as a proxy for the entire yield curve. That is, since the ten-year yield decreased by 7.5bp, the entire yield curve was assumed to have shifted down in parallel by 7.5bp. However, as indicated in Figure 4, the yield curve did experience a small degree of curve reshaping.

Figure 4. Yield-Curve Change, 7 Jun 96-27 Jun 96

Treasury Date	3M o	6Mo	1Yr	2Yr	3Yr	4Yr	5Yr	7Yr	10Yr	20Yr	30Yr
June 7, 1996	5.2 6	5.52	5.83	6.35	6.52	6.62	6.72	6.79	6.91	6.97	7.03
June 27, 1996	5.19	5.41	5.71	6.23	6.41	6.50	6.60	6.69	6.83	6.91	6.99
Yield Change (bp):	**-7**	**-12**	**-12**	**-12**	**-12**	**-12**	**-12**	**-10**	**-8**	**-6**	**-4**

bp Basis points.
Source: Salomon Brothers Inc.

With hindsight, we can determine the portion of the actual price change attributable to yield-curve changes. If we use the yield curve from June 27 to price the pass-through on June 7, keeping everything else unchanged, we obtain a price of 98-11, giving a change of +14 ticks. If we shift the curve in parallel by -7.5bp (the exact amount the ten-year changed by), we obtain a price change of +12 ticks. Hence, yield-curve reshaping accounts for roughly 2 ticks of the discrepancy.

Constant OAS Pricing

The OAS of the FNMA 7.5s widened from 57bp to 62bp between the two dates, while the effective duration calculation assumes an unchanged OAS. Repricing the pass-through on June 7 with an OAS of 62bp rather than 57bp gives a price 8 ticks lower than the constant OAS price. Thus, the constant OAS assumption accounts for -8 ticks of the discrepancy.

Constant Term Structure of Volatility

As shown in Figure 5, volatilities changed by as much as 0.6% between June 7 and June 27.

Figure 5. Change in Implied Volatility Levels, 7 Jun 96-27 Jun 96

	Ed Fut	1 Yr Cap	2 Yr Cap	3 Yr Cap	4 Yr Cap	5 Yr Cap	6 Yr Cap	10 Yr Cap	1x10 Swap	5x10 Swap
Treasury Date										
June 7, 1996	15.3	16.4	19.1	19.7	19.5	19.2	18.5	17.3	15.5	13.0
June 27, 1996	15.0	16.5	19.2	19.6	19.3	18.9	18.0	16.7	14.9	13.1
Volatility Change:	**-0.3**	**0.1**	**0.1**	**-0.1**	**-0.2**	**-0.3**	**-0.5**	**-0.6**	**-0.6**	**0.1**

Source: Salomon Brothers Inc.

Repricing the pass-through using the volatility levels from June 27 leads to an increase of about two ticks, indicating that volatility changes accounted for roughly 2 ticks of the discrepancy.[2]

Constant Mortgage-Treasury Spreads

Another assumption in the effective duration calculation is that current-coupon mortgage/Treasury spreads remain unchanged. However, as shown in Figure 6, current-coupon spreads in fact widened by 2bp between June 7 and June 27.

Figure 6. Change in Current-Coupon Spreads, 7 Jun 96-27 Jun 96

Treasury Date	MBS Current-Coupon Yield	10Yr. Yield	Spread (bp)
June 7, 1996	7.969	6.905	106
June 27, 1996	7.913	6.830	108
Current-Coupon Spread Change (bp)			**+2**

bp Basis points.
Source: Salomon Brothers Inc.

Repricing the pass-through for June 7 with current-coupon spreads 2bp wider leads to an increase of about 0.5 ticks.

Convexity and Asymmetric Price Movements

Effective duration is in essence obtained by averaging projected price changes when rates move up and down. If price changes are asymmetric, then effective duration will tend to overproject or underproject the changes. In many cases, MBSs have negative convexity,[3] which means that, other things being equal, effective duration will overproject price increases when rates move down and underproject price declines when rates move up.

If we define D^+ and D^- to be the durations when rates move up and when rates move down, respectively, then

$$\text{Effective Duration} = D = 0.5 * (D^- + D^+)$$

Hence, if with the benefit of hindsight we knew that rates were going to move down, we would use D^- for projecting the price move. The difference versus using effective duration is

$$D^- - 0.5 * (D^- + D^+) = 0.5 * (D^- - D^+)$$

$$\cong 0.5 * \Delta y * \mathit{Convexity}$$

Hence, the difference in projected prices is approximately

$$\mathit{Price} * \Delta y * (0.5 * \Delta y * \mathit{Convexity}) = \mathit{Price} * 0.5 * (\Delta y)^2 * \mathit{Convexity}$$

[2] The OASs quoted here are based on market volatilities, which are updated every day. If we were using fixed volatilities, then by definition volatilities in the model do not change, and the effect of changes in market volatilities, insofar as they were reflected in MBS prices, would show up as changes in OAS.

[3] For a discussion of MBS convexity and how it is calculated, see *Guide to Mortgage-Backed Securities*, Lakhbir Hayre, Salomon Brothers Inc., November 1995.

In our example, the price is 97-29, Δy is -7.5bp, and the convexity is -0.93, so that the error is

$$(97\text{-}29) * (-7.5\text{bp})^2 * 0.5 * (-0.93\%)$$

or about -0.1 ticks. Hence, convexity is not a significant factor in this case.[4]

Figure 7 summarizes the effects of the various assumptions. As indicated, OAS widening during the period was the most important factor.

Figure 7. Price Discrepancy Components for the FNMA 7.5% Pass-Through, 7 Jun 96-27 Jun 96

Actual Price Change (ticks)	+6
Projected Price Change Using Effective Duration	+12
Discrepancy	-6
	Discrepancy Components (ticks)
Assumption	
Parallel Yield-Curve Shifts	2
Constant OAS	-8
Constant Volatilities	2
Constant Mortgage Spread	0.5
Convexity	-0.1

OAS Option-adjusted spread.
Source: Salomon Brothers Inc.

Time Value, or the Cost of Carry

The discrepancy components in Figure 7 add to roughly -4 ticks, compared to the actual -6 tick discrepancy. The remainder is due mostly to differences in carry adjustments between the two dates. Although we are dealing with TBAs, rather than an actual bond, the time factor can still be important. The prices shown in Figure 3 are for forward settlement; the June 7 price is for settlement in June (typically in mid-June, on a date specified by the Public Securities Association (PSA)), while the June 27 price is for July settlement (in mid-July). For OAS calculations, the prices shown in Figure 3 are adjusted for next-day settlement (based on the drop, or differences in prices between June and July settlement). Because June 27 is about ten days further from the next PSA settlement date (in mid-July) than is June 7 (in mid-June), the adjustment for June 27 is roughly two ticks higher than for June 7. Hence, the change in *carry-adjusted* prices between the two dates was about 8 ticks, compared to the 6 tick change in the nominal prices. Thus, once we use carry-adjusted prices, the discrepancy between actual and projected drops to roughly 4 ticks, close to the sum of the components in Figure 7.

For periods of a few days or less, the time factor will usually not be important. Even for longer periods, the impact of time for TBAs depends typically not on the difference between the two dates but on the time from each date to the next PSA settlement date. Because of these considerations, we have not explicitly included time as one of the risk factors in Appendix A, but it is something investors should keep in mind. Using carry-adjusted prices will typically remove most of the effect of time.

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[4] Obviously, this is because the yield change (-7.5bp) is small; convexity would become more of an issue for larger market moves. A related issue is that effective durations are calculated for a chosen rate move (25bp in this case), and projected price moves for different rate moves (7.5bp in our example) are obtained by linear interpolation or extrapolation. However, this is typically not a problem except for very large rate moves or for some volatile MBS derivatives.

A General Attribution Formula for Mishedges

Appendix A presents a general approach for projecting price changes using durations and convexities to various risk factors. In particular, Eq. (A9), recast below, expresses the difference between actual and projected price changes in terms of these risk factors:

$$Actual\ -Projected\ Price\ Change = \Delta P - \Delta\hat{P}$$

$$\cong P\left[-D_s\Delta s - D_v\Delta v - D_c\Delta c + \frac{1}{2}C_y\Delta y^2 - \sum D_{y_j}(\Delta y_j - \Delta y)\right] \qquad (2)$$

where D_k and C_k represent duration and convexity to risk factor k respectively, and s = OAS, v = volatility, c = current-coupon spread, y = chosen Treasury yield (the ten-year in our examples) and y_j = key yield-curve rates.

Figure 8 summarizes the calculation of this equation for the FNMA 7.5% pass-through using four key yield-curve rates. For volatility, we take the average change of the volatilities shown in Figure 5, which gives a result close to the exact number.[5] Eq.(2) does not include adjustments for carry differences, which as discussed earlier account for about 2 ticks of the discrepancy.

Figure 8. Attribution of Difference Between Actual and Projected Price Changes

Risk Factor	Component	%Price Change	Actual Price Ch	Ticks
1) OAS	$-D_s\Delta s = -(5.1\%)(5bp)$	$= -0.255\%$	$\Rightarrow -0.250$	= -8.0
2) Volatility	$-D_v\Delta v = -(0.29\%)(-0.23)$	$= +0.067\%$	$\Rightarrow +0.065$	= +2.1
3) Current-Coupon Spread	$-D_c\Delta c = -(-0.83\%)(1.9bp)$	$= +0.016\%$	$\Rightarrow +0.015$	= +0.5
4) Convexity	$\frac{1}{2}C_y\Delta y^2 = \frac{1}{2}(-0.93\%)(7.5bp)^2$	$= -0.003\%$	$\Rightarrow -0.003$	= -0.1
5) Curve Reshaping	$-\sum D_{y_j}(\Delta y_j - \Delta y)$			
	$= -D_{2yr}(\Delta 2yr - \Delta y) = -(0.3\%)(-4bp)$			
	$- D_{5yr}(\Delta 5yr - \Delta y) = -(1.7\%)(-4bp)$			
	$- D_{10yr}(\Delta 10yr - \Delta y) = -(2.4\%)(0bp)$			
	$- D_{30yr}(\Delta 30yr - \Delta y) = -(0.6\%)(+4bp)$			
	=	$+0.056\%$	$\Rightarrow +0.055$	= +1.8

bp Basis points. OAS Option-adjusted spread.
Source: Salomon Brothers Inc.

The numbers shown in Figure 8 are, as might be expected, fairly close to the one-at-a-time manually calculated ones shown in Figure 7. Thus, Eq. (2) gives a convenient method for attribution of discrepancies between actual and projected price changes.

[5] A more precise method would be to use partial volatility durations, analogous to what we do for Treasury yield changes.

Summary

Our comparison of projected and actual price changes for two time periods suggests that effective duration will predict price movements well in some cases but not so well in others, depending on whether the underlying assumptions hold. How do we validate the predictive ability of the effective durations from a particular model? One way is to compare them to empirical durations, which are statistical estimates obtained by comparing actual price and yield changes. The next section explores various methods for calculating empirical durations and discusses their statistical characteristics.

Empirical durations refer to estimates of MBS price elasticity with respect to Treasury rates obtained from market data. While there are many possible ways of obtaining such measures, the standard approach involves regressing percentage MBS price changes against corresponding Treasury yield changes. We describe this method in more detail below, discuss what information it provides, and derive a relationship between empirical and effective durations. This relationship is used to derive an "updated" empirical duration, which combines the effective duration with the pertinent information provided by the empirical duration. Also discussed are alternative methods of calculating empirical durations, including those based on a fixed relative coupon (or constant dollar price); however, as we show, these other measures have their own limitations.[6]

Obtaining Empirical Duration Estimates

The usual method for calculating empirical durations is to regress daily MBS percentage price changes against corresponding yield changes for a benchmark Treasury (typically the ten-year). If P denotes MBS price and y the Treasury yield, then by definition,

$$dP/P = -\text{Duration} * dy \tag{3}$$

If $\Delta P/P$ and Δy are the actual price and yield changes on a given day, then based on Eq. (3) we assume that

$$\Delta P/P = \alpha - \beta * \Delta y + \text{noise term} \tag{4}$$

where β is the "true" duration, and α is a constant term. Given data ($\Delta P/P$, Δy) for a number of days, then standard regression methods can be used to obtain an estimate for β (see Eq. (B2) in Appendix B); this estimate, $\hat{\beta}$, say, is taken to be the empirical duration for the period.[7]

The Relationship Between Empirical and Effective Durations

The price of an MBS will depend on a number of factors: various points on the yield curve, volatilities, the OAS, and so on. Appendix B derives an expression for the empirical duration estimate β obtained using Eq. (4) in terms of the effective duration β and these various risk factors. If s denotes OAS, v denotes volatility,[8] and so on, then as shown in Appendix B,

$$\hat{\beta} \cong \beta + \mu + D_s * \text{Corr}(\Delta s, \Delta y) * \text{Vol}(\Delta s)/\text{Vol}(\Delta y)$$
$$+ D_v * \text{Corr}(\Delta v, \Delta y) * \text{Vol}(\Delta v)/\text{Vol}(\Delta y) + \ldots \tag{5}$$

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6 Earlier work on empirical durations includes papers by Pinkus & Chandoha, (*Journal of Portfolio Management*, Summer 1986), DeRosa, Goodman and Zazzarino, (*Journal of Portfolio Management*, Winter 1993), and Breeden (*Journal of Fixed Income*, September 1991 and December 1994). The focus of these papers is measuring market durations and (in the Breeden papers) on their hedging effectiveness, whereas ours is on exploring the theoretical relationships between empirical and effective durations.

7 Why is an intercept term used in Eq. (4)? In other words, why not use

$$\Delta P/P = -\beta * \Delta y + \text{noise term}$$

to estimate the duration? The reason is that having an intercept term "detrends" the data, so that the estimate for β is not distorted through having to incorporate price changes unrelated to yield changes. In practical terms, it typically makes little difference as to whether an intercept term is used or not.

8 For ease of notation, we will assume just one volatility, although our formulation allows us to include as many volatilities (and other risk factors) as desired.

where $\hat{\beta}$ = empirical duration estimate

β = current effective duration

μ = average difference between current effective duration and the effective durations over time period used for the data

D_k = duration of MBS with respect to risk factor k

$Corr(\Delta k, \Delta y)$ = sample correlation between changes in risk factor k and changes in y over the sample time period

$Vol(\Delta U)$ = sample standard deviation (or volatility) of daily changes in variable U over the sample time period.

In practice, the most important factor is a change in OAS. If we ignore other risk factors, ignore the effect of noise *and assume that the duration is fairly stable over the time period used*, then, approximately,

$$\text{Emp Dur Estimate} = \hat{\beta} \cong \beta + D_s * \text{Corr}(\Delta s, \Delta y) * \text{Vol}(\Delta s)/\text{Vol}(\Delta y) \qquad (6)$$

where D_s is the OAS duration of the MBS, Corr (Δs, Δy) is the correlation between OAS and Treasury yield changes over the time period used, and Vol (Δs) and Vol (Δy) denote the standard deviation of Δs and Δy, respectively, over the sample time period.

Why Effective Durations Are Often Longer Than Empiricals

Eq. (6) shows why empirical durations are often shorter than effective durations. If there is significant directionality between daily OAS and yield changes, with a negative correlation between them (so that a drop in yield leads to widening in OAS), then the empirical duration will be shorter than the effective duration. **This will be true even if there is no net change in OAS over the period, and the cumulative price change is in line with that predicted by effective duration.**

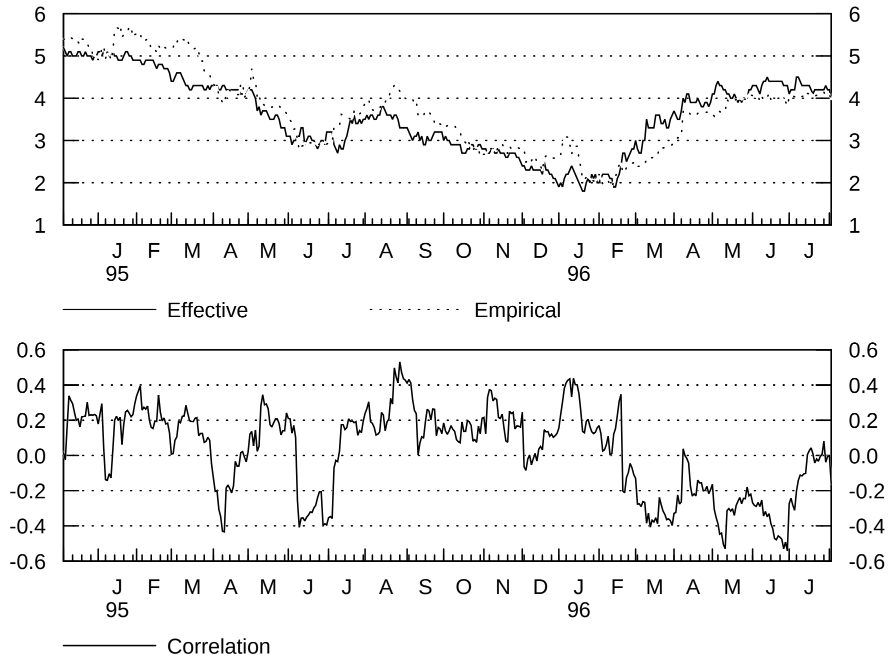
As a numerical example, for the period June 14, 1996 to July 16, 1996, GNMA 8.5s had an empirical duration of 3.2, while the effective duration averaged around 4.0, and the OAS duration was 4.5. The correlation between OAS and ten-year Treasury yield changes for this period was -0.73, while the daily standard deviations were 0.023 and 0.078, respectively. From Eq. (6), this implies that the effective duration should be longer than the empirical by approximately

$$4.5 * (0.73) * \left(\frac{0.023}{0.078}\right) = 0.97$$

which is relatively close to the actual difference of 0.8.

Empirical and effective durations will tend to diverge when there is a high correlation between OAS and yield changes, which tends to occur during periods when there is a high degree of prepayment fears. This is illustrated in Figure 9, which shows historical correlations between ten-year Treasury yield and OAS changes for conventional 8s, along with effective and empirical durations; both the empirical durations and correlations are based on the previous month of data at each point.

Figure 9. Effective and Empirical Durations for Conventional 8s (top) and Correlations Between Ten-Year Treasury Yield Changes and Changes in OAS (bottom)



bp Basis points. OAS Option-adjusted spread.
Source: Salomon Brothers Inc.

Our studies indicate that the adjustment shown in Eq. (6) usually explains most of the difference between empirical and effective durations.

When Are Empirical Durations Better for Predicting Price Changes?

Later in this paper, we describe the results of historical studies comparing the hedging efficiencies of empirical and effective durations. We note here that the choice between using empirical or effective durations for hedging involves making fundamentally different assumptions about the relationship between past and future price movements. Eq. (A9) in Appendix A gives the difference between predicted and actual price changes if we use effective duration, while Eq. (B7) in Appendix B gives the corresponding difference if we use empirical duration.

• If effective duration is used, the prediction error is due to changes in risk factors such as OAS, volatilities and current-coupon spreads, to yield-curve reshaping, and to second order (convexity) effects.

• If empirical duration is used, the prediction error is due to the relationships between yield changes and changes in the risk factors differing from those implied by historical data (where historical means over the period used to calculate the empirical duration), and also due to shortening or lengthening of duration over the period (this is captured by the μ term).

Thus, it is better to use effective duration if we believe that, for example, correlations between OAS and yield changes do not generally show any systematic pattern (i.e., our OAS model does not on average display any rate dependence) and hence that we should not try to predict such correlations. Even if daily OAS and yield changes do show a correlation, changes over a week or a month may not, and hence effective duration may still be better unless reducing day-to-day fluctuations in our position is of critical importance. Thus, empirical durations will be better only if we believe that past correlations between Treasury yield changes and the various risk factors will repeat themselves going forward (for example, if the OASs from our model show a systematic and predictable rate dependence). *However, in this latter case, it is preferable to use the updated empirical duration defined next, to take account of substantial market moves over the time period used for empirical duration calculations.*

Combining Empirical and Effective Durations

Investors who lean toward empirical durations should instead use an adjusted version derived from Eq. (5) above. We define this as

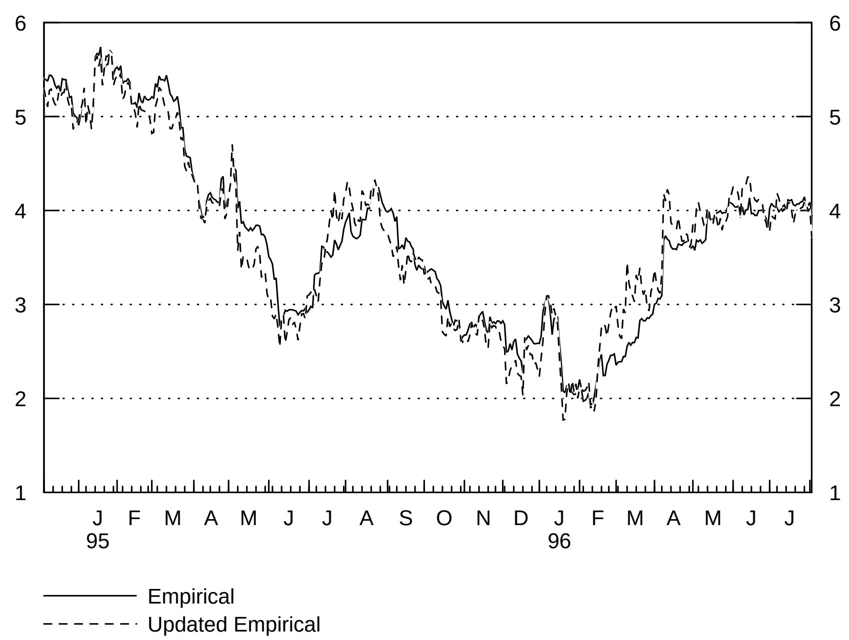
$$\text{Updated Emp Dur} = \text{Emp Dur} - \mu \cong \beta + D_s * \text{Corr}(\Delta s, \Delta y) * \text{Vol}(\Delta s) / \text{Vol}(\Delta y)$$

$$+ D_v * \text{Corr}(\Delta v, \Delta y) * \text{Vol}(\Delta v) / \text{Vol}(\Delta y) + \ldots\ldots \tag{7}$$

This is in effect equivalent to the empirical duration adjusted for duration changes over the sample time period. In other words, the updated empirical duration incorporates the information provided by empirical duration and also uses current market information, as captured by the effective duration. *It can alternatively be thought of as the effective duration adjusted for the correlations between changes in the yield and changes in risk factors displayed by recent market data.*

Figure 10 shows historical values for the empirical duration and for the updated empirical duration for conventional 8s.

Figure 10. Empirical and Updated Empirical Durations for Conventional 8s



Source: Salomon Brothers Inc.

The updated empirical duration is generally very close to the empirical; any differences between the two reflect the effect of recent market moves that can make the empirical durations out of date. For example, in the spring of 1995, when rates were falling, the updated empirical duration declined faster; similarly, a year later, when rates were rising, the updated empirical duration rose faster.

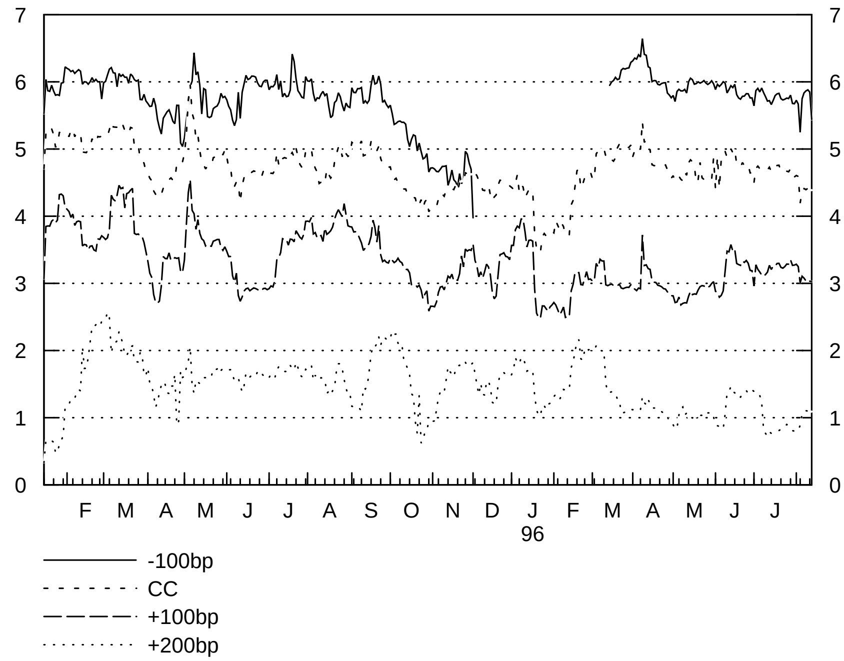
A hedging note. At the risk of stating the obvious, note that neither effective nor empirical durations will lead to a hedge against price moves that are *uncorrelated* with yield moves. For example, if the OAS widens or tightens in a way unrelated to Treasury yield changes, then Treasuries will not provide a hedge for the MBS price change resulting from the OAS move.

Constant Relative Coupon (or Constant Price) Durations

MBS durations change with interest rates, so that if rates have moved substantially, the empirical duration for a given coupon can be a poor indicator of the likely duration going forward. This had led to the development of empirical durations for a fixed relative coupon (or, more or less equivalently, for a fixed dollar price), where we estimate the empirical duration not for a fixed coupon (say 8s), but a fixed relative coupon (for example, the current coupon). Thus, the price moves used in the calculation may not (and typically will not) be for the same MBS over the whole time period. For example, if we are calculating the empirical duration for the current coupon, then for each day, the price move will be for the MBS that was the current coupon on that particular day.

Though empirical durations by relative coupon can provide valuable information, there can be problems with this solution to a real problem (durations changing over the time). The first and obvious one is that different MBSs may differ in key features, such as WAMs, previous prepayment history, etc., and hence will not display the same durations even when they are the same relative coupon. Second, as the last several years should have taught us, prepayments, and hence durations, depend not just on the relative coupon but also on the absolute level of rates. Thus, even for the same relative coupon, durations can change substantially over time, as illustrated in Figure 11.

Figure 11. Empirical Durations by Relative Coupon for Conventional TBAs



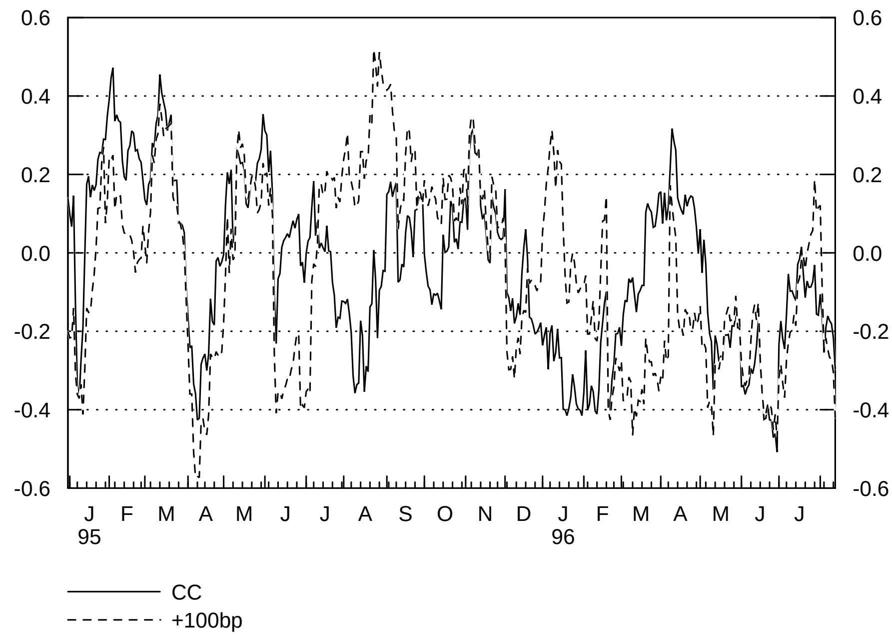
bp Basis points. CC Current coupon.
Source: Salomon Brothers Inc.

Even for discount relative coupons, the duration can change by more than half a year in a single month, and for a cuspy coupon (such as current coupon plus 200bp), the duration has sometimes changed by a factor of two or more in a single month.

A practical problem with relative coupon durations is that available data may be suspect or may not even exist. For example, in the spring of 1995, after rates started falling sharply, the "+200bp" durations were based on price moves of 10s and higher coupons, which tend to be illiquid. The break in the "-100bp" line in Figure 11 at the end of 1995 and early 1996 means that there were no TBA coupons 100bp below the current coupon.

Is there more systematic directionality (i.e., more persistent correlations) between OAS and yield changes for a fixed relative coupon? The data suggests not. Figure 12 shows correlations between changes in ten-year Treasury yields and changes in OASs for conventional current coupons and for current coupons +100bp.

Figure 12. Correlations Between Ten-Year Treasury Yield Changes and Changes in OASs by Relative Coupon



bp Basis points. CC Current coupon.
Source: Salomon Brothers Inc.

There is little difference in the two time series in Figure 12, and both seem equally difficult to predict.

The updated empirical duration defined above provides, in our opinion, a more scientific method (one that is based on the statistical properties of empirical duration estimates) for dealing with the problem of durations changing with interest rates. It uses current market information (via the effective duration), as well as distilling relevant information provided by recent market data.

Empirical Durations Based on Price Levels, Not Price Changes

An implicit assumption in empirical duration calculations is that the Treasury yield change on a given day has an impact on the MBS price that same day; this is what is expressed, for example, by Eq. (4) above. While this is a reasonable assumption for liquid, actively traded securities, it may not be true in other cases.

An example is provided by high premium pass-throughs. The float on these MBSs is small, much of the trading is on a specified pool basis, and hence there is not much of a TBA market. As a result, prices for high premium TBAs often react with a lag, responding cumulatively to several days worth of Treasury curve changes. Hence, comparing daily price changes with corresponding Treasury yield changes suggests very little relationship, leading to a low estimate for empirical duration.

An alternative approach, described in an earlier article,[9] is to compare price levels with yield levels. That is, rather than regressing $\Delta P/P$ versus Δy, we can,

[9] See *Bond Market Roundup: Strategy*, November 3, 1995, Salomon Brothers Inc.

for example, regress log P versus y.[10] The negative of the slope will be the empirical duration estimate.

It is important to be clear as to what this empirical duration estimate measures. It describes the relationship between MBS price and Treasury yield *levels* over a period of time, rather than the relationship between day-to-day *changes*; hence it may lead to a poor hedge against daily yield-curve fluctuations. Even for it to be useful in deriving a long-term hedge, it has to be assumed that changes in OAS (and other risk factors) that occurred during the historical period would be repeated over the time period for the hedge.

Partial Empirical Durations

Partial durations can be used to hedge against yield-curve reshaping. We can estimate empirical partial durations by using a multiple regression version of Eq. (4):

$$\Delta P/P = \beta_0 - \beta_1 {*} \Delta y_1 - .... - \beta_j {*} \Delta y_j \ + \ \text{noise term}$$

for selected Treasury yields $y_1, \ldots, y_j$. However, Treasury yields of different maturities tend to be highly correlated, leading to regression estimates for the βs that can be unstable. Given the complexity and scope of this topic, we will leave a discussion of partial durations (effective or empirical) to a future paper.

Summary

The topic of empirical durations, like mortgage durations in general, is shrouded in confusion. Many analysts, traders, and investors have turned to empirical durations out of frustration with effective durations, and use empiricals either for determining hedge ratios or for making ad-hoc adjustments to effective durations. While empirical durations can provide valuable information about recent market relationships, making intelligent use of this information requires an understanding of the statistical properties of the empirical duration estimates. The updated empirical duration defined here incorporates all the relevant information provided by both effective and empirical durations, and should be used if it is believed that directionality in OASs is a feature of the model being used (i.e., the OASs display a persistent rate dependence). However, as we show in the next section, if the OASs from a particular model (such as Salomon's) show no sustained and systematic rate dependence, then over the long term it is better to use effective durations.

[10] Log P versus y is preferable to P versus y, since if log P = a + by, then [dP/dy]/P = -Duration = b.

The ultimate test of any duration estimate is its performance in constructing effective hedges. This section describes the results of a historical study of hedging performance using effective durations, empirical durations, and the empirical/effective combination we have labeled updated empirical duration.

Our study is not meant to be comprehensive (in terms of MBSs and time periods used),[11] and uses a simplified hedge (only one Treasury, no cash flows, no financing considerations, etc.). However, it provides some insight into the relative hedging efficiencies of empirical and effective durations.

Reasons for Hedging Errors

Assuming that there are no cash flows during the period, then the change in portfolio value is simply

Hedging Error = MBS Price Change - Hedge Ratio * Treasury Price Change

Eqs. (A9) and (B7) in the appendices give general expressions for the hedging errors using effective and empirical durations, respectively. The following items reiterate the points made in the previous section:

• **Effective duration** assumes that OASs and other risk factors are unchanged, and hence hedging errors will be due to changes in OAS and other risk factors.

• **Empirical duration** assumes that past relationships (such as correlations) between changes in OAS and other risk factors and changes in the yield of the Treasury being used will hold going forward; in other words, for a given yield change, that the OAS and other factors will change by amounts implied by past patterns (where past means the time period over which the empirical duration is calculated). Hence, hedging errors will be due to OASs and other risk factors changing by amounts different than those implied by past data. In addition, hedging errors may arise due to the duration changing over the sample time period.

• **Updated empirical duration** modifies the empirical duration to remove the effect of recent duration changes.

Directionality in OASs

As the previous discussion makes clear, the key assumption behind effective duration is that OASs remain unchanged. If there is a systematic dependence of OASs on rate levels -- for example, OASs being higher on higher coupons -- then effective durations will show a systematic bias. For Salomon's model, OASs have tended to be relatively flat across coupons for liquid TBA pass-throughs, as Figure 13 indicates.[12]

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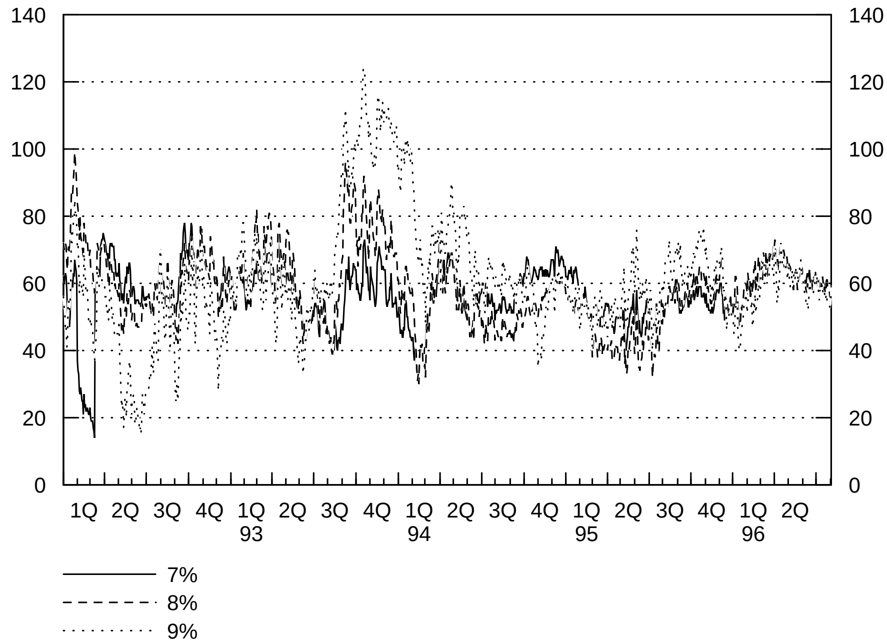
[11] We have restricted attention to liquid TBA coupons, taken to be 6.5s through 9s, and the time period studied is approximately the last two years, both because it can be considered the "modern" era, when the lessons of 1991-1993 have been hopefully absorbed, and because it represents roughly a year of "within sample" and a year of "out-of-sample" results for Salomon's MBS models (which were implemented in the summer of 1995).

[12] For readers with access to Salomon's *Yield Book* system, manifold MB725 shows current and historical OASs. There has recently developed a slight upward tilt to OASs on GNMAs, possibly due to fears about new developments such as the popularity of the streamlined refinancing program. In addition, OASs on most coupons widened dramatically in late 1993, with higher coupons widening the most. However, this widening was not repeated in 1995 and 1996, suggesting a more mature market.

Figure 13. OASs on Conventional TBAs, Jan 92-Aug 96



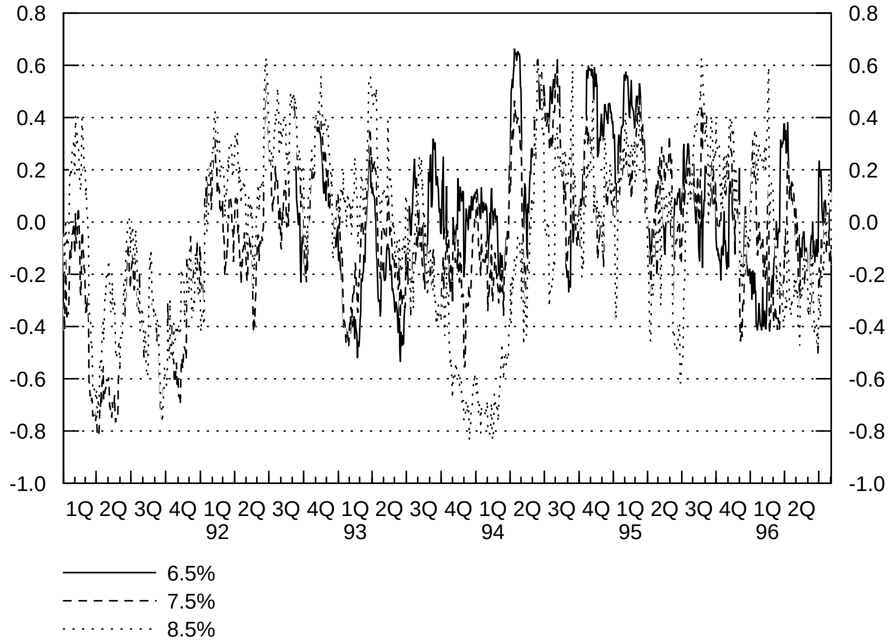
Source: Salomon Brothers Inc.

If OASs tend to be fairly stable across time and across coupons, then it follows (if we neglect other factors, such as volatility) that effective durations should work well *on average over time.* A recent Salomon Brothers study[13] confirmed this, showing that duration-adjusted monthly returns of MBSs, using effective durations from Salomon's models, were fairly unbiased versus changes in the ten-year Treasury.

What about shorter horizons, such as a day? Figures 9 and Figure 14 suggest there is often significant directionality between OAS and Treasury yield changes.

13 See the article by Bob Kulason in *Bond Market Roundup: Strategy*, July 19, 1996, Salomon Brothers Inc..

Figure 14. Rolling One-Month Correlations Between Ten-Year Treasury Yield Changes and Changes in the OASs of Conventional TBAs, Jan 91-Aug 96



Source: Salomon Brothers Inc.

Empirical durations (or what is arguably an improved version, the updated empirical duration) will tend to do better than effective durations if there is systematic and predictable correlations between changes in Treasury yields and changes in risk factors, such as OAS. However, although there are periods (such as late 1993 and the spring of 1996) when the correlations were persistently negative (which would result in empirical durations being shorter than effectives), there are periods when they are positive, and the time series in Figure 14 does not suggest much predictability. As the hedging results below indicate, *the key issue is not whether OAS and yield changes are correlated, but how stable and predictable these correlations are.*

Historical Performance

Figure 15 shows the results of a historical study of hedging performance for conventional TBA coupons. In the study, each MBS is hedged with the ten-year Treasury, with the hedge ratio calculated from the respective durations, and the hedge is rebalanced every day. For the empirical and updated empirical durations, the last month of data is used for calculations. Hedging efficiency is measured by the standard deviations of the hedging errors.

Figure 15. Historical Hedging Efficiency of Empirical and Effective Duration Measures for Conventional TBA Coupons, Dec 94-Aug 96

	Duration Measure Used			
Issue	Unhedged	Empirical	Updated Empirical	Effective
6.5%	$3379.37	$758.43	$752.67	$750.06
7.0	3134.48	727.20	720.61	702.36
7.5	2845.32	729.74	721.36	695.43
8.0	2437.96	712.13	695.30	674.04
8.5	2053.59	752.67	747.70	724.87
9.0	1597.47	818.17	824.47	829.70

Note: Numbers shown are standard deviations of daily changes for $1mm face of MBSs.
Source: Salomon Brothers Inc.

The effective duration does better in most cases. In other words, trying to predict changes in OASs and other risk factors using recent correlations with Treasury yield changes (as empirical duration in essence does) generally leads to sub-par hedging performance. Better predictions may be necessary for MBSs that are very prepayment-sensitive, indicated by the fact that the only coupon for which empirical durations were slightly better is the premium 9%.[14] However, we also note that if we used weekly time periods rather than daily, the effective durations did better than empiricals even for the 9s, confirming that directionality in OASs becomes less of a problem with longer hedging horizons.

The updated empirical duration generally does better than the empirical. This should not be surprising, since the updated empirical duration factors in recent market changes.

It is instructive to take a closer look at a particular coupon. Figure 16 shows a month-by-month breakdown of the hedging performance of conventional 8s. Also shown are the empirical and average effective durations, and the correlations between OAS and ten-year yield changes, for each month.

Figure 16. Month-by-Month Hedging Performance for Conventional 8s

						Duration Measure Used		
	Yield Change	Corr.	Emp. Dur.	Avg. Eff. Dur.	Unhedged	Emp. Dur.	Upd. Emp. Dur.	Eff. Dur.
Dec-94	0.0 7	0.24	5.01	5.04	$2336.51	$842.01	$842.16	$826.57
Jan-95	-0.15	0.37	5.49	4.99	3133.18	895.14	879.53	853.45
Feb-95	-0.43	0.01	5.20	4.76	2992.96	495.83	490.42	480.37
Mar-95	-0.16	-0.20	4.27	4.34	2583.12	678.84	642.33	602.40
Apr-95	-0.06	0.12	4.28	4.20	1138.90	498.65	498.07	478.78
May-95	-0.78	0.21	3.48	3.57	2462.58	870.86	809.64	790.45
Jun-95	0.11	-0.35	2.98	3.04	2696.43	909.00	856.80	839.62
Jul-95	0.29	0.27	3.92	3.22	2848.81	1026.39	1013.43	1032.08
Aug-95	-0.18	0.43	4.00	3.52	2164.35	598.44	575.86	610.10
Sep-95	-0.04	0.14	3.33	3.09	1717.85	544.84	522.88	523.09
Oct-95	-0.14	0.12	2.68	2.85	1153.63	499.87	491.70	469.80
Nov-95	-0.15	-0.07	2.79	2.66	1300.77	466.83	466.64	459.63
Dec-95	-0.05	0.23	2.66	2.23	1367.16	752.63	758.32	738.04
Jan-96	0.03	0.13	2.10	2.09	1593.02	912.20	910.57	849.70
Feb-96	0.35	-0.28	2.39	2.41	1887.15	602.29	661.70	589.02
Mar-96	0.37	-0.32	3.01	3.34	3088.98	727.06	585.10	540.44
Apr-95	0.45	-0.31	3.58	3.89	3071.53	651.91	668.83	571.98
May-96	-0.03	-0.28	4.04	4.13	2829.55	604.71	570.36	552.53
Jun-96	-0.11	-0.24	4.08	4.33	2707.84	737.07	757.45	730.34
Jul-96	-0.16	-0.16	3.98	4.25	3503.96	540.55	522.48	554.90

Notes: (1) Empirical duration and correlations based on observations for the calendar month. (2) The "Avg. Eff. Dur." is the average of the daily effective duration for the month.
Source: Salomon Brothers Inc.

There are several points worth noting from Figure 16:

• Effective duration does best in most months but given the erratic nature of actual MBS price moves, there are several months where empirically based measures performed best.

[14] Better predictions may be possible if other information, such as measures of refinancing activity or prepayment fear (for example, the "media effect") are also used. However, this is outside the scope of the present publication.

• There seems little relationship between the degree of correlation between OAS and yield changes and the relative performance of effective and empirical durations. For example, during most of 1996 to date, the correlations have been negative, leading to effective durations being longer than empiricals, yet the effective duration has generally led to better hedges.

Perhaps the most striking aspect of the numbers in Figures 15 and 16 is the quite small difference between using effective and empirical durations, a point also noted in earlier studies.[15] Both reduce risk (as measured by fluctuations in portfolio value) by quite a substantial amount, but neither reduces it completely. Both methods will tend to fail when there are unexpected and significant changes in OASs; thus, as indicated earlier, the key to further risk reduction may be in trying to predict such changes.

Can we do better using constant relative coupon durations? The correlations shown previously in Figure 12 suggest not, and hedging performance confirms this. Although reliable relative coupon empirical durations are not always available, especially after sudden market moves (when the history may be for coupons that are not liquid or simply did not exist), hedging results were generally inferior to simply using empirical duration. For 8s, for example, a coupon for which reliable relative coupon durations were available for the whole time period studied, relative coupon durations underperformed standard empirical durations in the majority of months, and underperformed updated empirical durations in all but one month.[16] For 8.5s and 9s, the results were basically similar, although there were periods when the relative coupon durations were based on illiquid higher coupons and hence were not too meaningful. For lower coupons, relative coupon durations were not always available, making a comparative study difficult.

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[15] See Douglas T. Breeden, "Complexities of Hedging Mortgages," *Journal of Fixed Income*, December 1994
[16] The differences were typically small, though, being about the same order of magnitude as between effective and empirical durations.

No single duration measure will consistently work well for mortgage securities. MBS price changes are determined by a number of risk factors (such as OAS and volatility changes), which often change in unpredictable ways. All MBS duration measures have various and differing embedded assumptions about how these risk factors will change with Treasury yields, and hence a particular duration measure will be off whenever its assumptions are violated.

The first step in evaluating different duration measures is to determine what assumptions are made in each case. While the assumptions made in standard effective duration calculations (parallel yield-curve shift, unchanged OASs, etc.) are generally understood, there seems to be only a hazy idea about the assumptions that are being made when using empirical durations to project MBS price moves. As shown in this paper, while effective durations assume that risk factors such as OASs will remain unchanged, empirical durations assume that these factors will maintain the same relationship with yield changes that were displayed during the data time period over which the empirical duration was estimated.

Clearly, given these assumptions, there will be time periods when effective durations will work better, other periods when empiricals will work better, and still other periods when neither will work very well. Our historical analysis suggests that, assuming that the OASs from the model being used do not have a systematic dependence on levels (such as OASs increasing with coupon, as is in fact the case with many models), then on average the effective durations tend to be better; while OASs and other risk factors do change, trying to guess what these changes will be based on the recent past (as empirical durations do) more often than not turns out to give inferior results relative to just assuming that the changes will be zero (as effective duration does).[17]

Two final points. First, *it is not necessarily a cause for alarm if the effective durations from your model are longer than empirical durations.* This may just reflect directionality between day-to-day changes in yields and OASs, which may diminish with longer periods such as a week or a month, and hence would be a concern only if it was important to minimize daily fluctuations in your portfolio. Second, *if it is wished to combine effective and empirical durations, then using an ad-hoc scheme, such as taking a weighted average of the two, is not the right way to do it.* Once the statistical relationship between empirical and effective durations has been determined (as in Appendix B), then an intelligent combination of the two can be chosen (for example, what we have labeled the updated empirical duration).

[17] This does not mean that a good attempt cannot be made to model changes in OAS and other risk factors in terms of yield changes -- just that the model implicit in empirical durations typically does not work very well. Since no paper worth its salt will end without suggestions for further research, we suggest this area as a critical one for modelers to focus on.

Actual Price Move

Let k_1 through k_N be risk factors such as OAS, volatilities, yield-curve rates, etc. For given changes $\Delta k_1 ..., \Delta k_N$ in these risk factors, using a Taylor Series expansion, a mortgage security's price change can be expressed as

$$\Delta P = \sum_{j=1}^{N}\left(\frac{\partial P}{\partial k_j}\Delta k_j + \frac{1}{2}\frac{\partial^2 P}{\partial k_j^2}\Delta k_j^2\right) + \text{cross/higher order terms.}$$

Dividing by the original price P, the percentage change in price is given by

$$\frac{\Delta P}{P} = \frac{1}{P}\sum_{j=1}^{N}\left(\frac{\partial P}{\partial k_j}\Delta k_j + \frac{1}{2}\frac{\partial^2 P}{\partial k_j^2}\Delta k_j^2\right) + \text{cross/higher order terms} \quad \text{(A1)}$$

Defining partial duration with respect to *k* as $D_k = -\frac{1}{P}\frac{\partial P}{\partial k}$, and partial convexity with respect to *k* as

$C_k = \frac{1}{P}\frac{\partial^2 P}{\partial k^2}$, then (A1) becomes

$$\frac{\Delta P}{P} = \sum_{j=1}^{N}\left(-D_{k_j}\Delta k_j + \frac{1}{2}C_{k_j}\Delta k_j^2\right) + \text{cross/higher-order terms}$$

Limiting risks to OAS, a single volatility, current-coupon spread, and yield-curve risks,[18] and neglecting all higher-order terms except yield-curve convexity gives

$$\frac{\Delta P}{P} = -D_s\Delta s - D_v\Delta v - D_c\Delta c - \sum D_{y_j}\Delta y_j + \frac{1}{2}\sum C_{y_j}(\Delta y_j)^2 \quad \text{(A2)}$$

where s = OAS, v = volatility, c = current-coupon spread, and y_j = key yield-curve rates.

For a given yield-curve rate, *y*, say, let D_y and C_y be the effective duration and convexity. Note that, neglecting higher-order terms, $D_y = \sum D_{y_j}$ and $C_y = \sum C_{y_j}$. We can now rewrite (A2) as approximately

$$\frac{\Delta P}{P} = -D_s\Delta s - D_v\Delta v - D_c\Delta c - D_y\Delta y + \frac{1}{2}C_y\Delta y^2 - \sum D_{y_j}(\Delta y_j - \Delta y) \quad \text{(A3)}$$

where we have ignored terms involving $(\Delta y_j^2 - \Delta y^2)$. Note that the $(\Delta y_j - \Delta y) = \Delta(y_j - y)$ terms measure yield-curve reshaping.

[18] This is by no means a complete set of risk factors or durations; among others could be prepayment and time durations. The risk factors cited are generally the most important for typical MBSs.

Effective Duration

Effective duration assumes that the yield-curve shifts in parallel and other risk factors are unchanged. That is, $\Delta y_j \equiv \Delta y$, $\Delta s = 0$, $\Delta c = 0$, and $\Delta v = 0$. Eq. (A3) then becomes (ignoring higher-order terms)

$$\frac{\Delta P}{P} \cong -D_y \Delta y + \frac{1}{2} C_y \Delta y^2 \qquad \text{(A4)}$$

Assume that $\Delta y > 0$. Then if rates back up by Δy, (A4) gives

$$\frac{P^+ - P}{P} \cong -D_y \Delta y + \frac{1}{2} C_y \Delta y^2 \qquad \text{(A5)}$$

Similarly, if rates rally by Δy, (A4) gives

$$\frac{P^- - P}{P} \cong D_y \Delta y + \frac{1}{2} C_y \Delta y^2 \qquad \text{(A6)}$$

(A6) - (A5) gives

$$\frac{P^- - P^+}{P} \cong 2 D_y \Delta y$$

Effective duration is then given by

$$\left(\frac{P^- - P^+}{P}\right)\left(\frac{1}{2\Delta y}\right) \cong D_y = -\frac{1}{P}\frac{dP}{dy} \qquad \text{(A7)}$$

Hence, for a given yield change Δy, the projected percentage change in price using effective duration is given by

$$\frac{\Delta \hat{P}}{P} = -(Effective\ Duration)\Delta y \cong -D_y \Delta y \qquad \text{(A8)}$$

Difference Between Actual and Projected Price Changes

The difference between the actual percentage price change and that projected by effective duration is given approximately by (A3) - (A8), or

$$\frac{\Delta P}{P} - \frac{\Delta \hat{P}}{P} \cong -D_s \Delta s - D_v \Delta v - D_c \Delta c + \frac{1}{2} C_y \Delta y^2 - \sum D_{y_j} (\Delta y_j - \Delta y) \qquad \text{(A9)}$$

Linear Regression

Empirical durations are typically calculated using a linear regression model given by

$$Y_t = \alpha - \beta X_t + \varepsilon_t \qquad \text{(B1)}$$

where $Y_t = \frac{\Delta P}{P} = \frac{P_t - P_{t-1}}{P_{t-1}}$, or the proportional change in price, and $X_t = \Delta y_t = y_t - y_{t-1}$, or the daily yield change (e.g., for the ten-year Treasury).[19]

Least squares minimization leads to a slope estimator of

$$\hat{\beta} = -\frac{\sum Y_t X_t - \frac{1}{N}\sum Y_t \sum X_t}{\sum X_t^2 - \frac{1}{N}\left(\sum X_t\right)^2} \qquad \text{(B2)}$$

where N = number of observations. The empirical duration is then taken to be $\hat{\beta}$.

What Does Empirical Duration Measure?

Eq. (B1) assumes that α and β are constant. In fact, referring to Eq. (A3) in Appendix A and neglecting higher-order terms,

$$\frac{\Delta P}{P} \cong \left[-D_s \Delta s - D_v \Delta v - D_c \Delta c + \frac{1}{2} C_y \Delta y^2 - \sum D_{y_j} (\Delta y_j - \Delta y) \right] - D_y \Delta y \qquad \text{(B3)}$$

For day t, let α_t be the value of the term in brackets, and let $\beta_t = D_y$.

Hence, the true relationship is

$$Y_t = \alpha_t - \beta_t X_t + \varepsilon_t ,$$

where ε_t captures influences on $\frac{\Delta P}{P}$ other than those shown in Eq. (B3).

Substituting this Y_t into Eq. (B2), the numerator becomes

$$\begin{aligned}
&\sum Y_t X_t - \frac{1}{N}\sum Y_t \sum X_t \\
&= \sum (\alpha_t - \beta_t \Delta y_t + \varepsilon_t)\Delta y_t - \frac{1}{N}\sum (\alpha_t - \beta_t \Delta y_t + \varepsilon_t) \sum \Delta y_t \\
&= \sum \alpha_t \Delta y_t - \sum \beta_t (\Delta y_t)^2 + \sum \varepsilon_t \Delta y_t - \frac{1}{N}\sum \alpha_t \sum \Delta y_t + \frac{1}{N}\sum \beta_t \Delta y_t \sum \Delta y_t - \frac{1}{N}\sum \varepsilon_t \sum \Delta y_t
\end{aligned}$$

[19] Note a change in notation: in Appendix A, Δy_k denoted the change in the k^{th} yield-curve rate, whereas here Δy_t represents the change in a given yield-curve rate between times (t-1) and t.

Let β denote the current value of β_t (i.e., of $-\frac{1}{P}\frac{dP}{dy}$), and define $\mu_t = \beta_t - \beta$.

Without loss of generality, we can assume that the ε_t term is noise. We further assume that μ_t and Δy_t have a very low correlation.

In addition, if we define

$$\textit{sample covariance} = COV(A,B) = \frac{1}{N}\left[\sum\left(A_i B_i\right) - \frac{1}{N}\sum A_i \sum B_i\right]$$
$$\textit{sample variance} = VAR(A) = \frac{1}{N}\left[\sum A_i^2 - \frac{1}{N}\left(\sum A_i\right)^2\right],$$

then straightforward algebra shows that, approximately

$$\hat{\beta} = \beta + \mu - \frac{\mathrm{COV}(\alpha, \Delta \mathrm{y})}{\mathrm{VAR}(\Delta \mathrm{y})} + \mathrm{NOISE} \qquad \text{(B4)}$$

where μ is the average of μ_t over the sample period (i.e., it is the average difference between the current effective duration and the ones from the data period), and *NOISE* refers to the terms involving ε_t.

From Eq. (B3), $\alpha_t = -D_s \Delta s_t - D_v \Delta v_t - D_c \Delta c_t + \frac{1}{2} C_y (\Delta y_t)^2 + \ . \ . \ .$

Now, for any variables U and V,

$$\frac{COV(U,V)}{VAR(V)} = \rho_{UV} \frac{\sigma_U}{\sigma_V},$$

where $\rho_{UV} =$ correlation between U and V

$\sigma_U =$ standard deviation of U

$\sigma_V =$ standard deviation of V

Eq. (B4) can now be written as

$$\hat{\beta} \cong \beta + \mu + D_s \rho_{\Delta s \Delta y} \frac{\sigma_{\Delta s}}{\sigma_{\Delta y}} + D_v \rho_{\Delta v \Delta y} \frac{\sigma_{\Delta v}}{\sigma_{\Delta y}} + \ldots + NOISE \qquad \text{(B5)}$$

where $\rho_{\Delta s \Delta y}$, etc., are sample correlations.[20]

Prediction Error Using Empirical Duration

Eq. (A9) in Appendix A gave the difference between actual and projected prices using effective duration. The corresponding error using empirical duration is

[20] For simplicity, we have assumed that D_S, D_V, ... are relatively constant over the sample period. In general, under fairly reasonable assumptions, Eq. (B5) will hold with D_S, ... replaced by sample averages.

$$\frac{\Delta P}{P} - (-\hat{\beta}\Delta y) \cong \mu\Delta y + D_s\left[\rho_{\Delta s\Delta y}\frac{\sigma_{\Delta s}}{\sigma_{\Delta y}}\Delta y - \Delta s\right] + D_v\left[\rho_{\Delta v\Delta y}\frac{\sigma_{\Delta v}}{\sigma_{\Delta y}}\Delta y - \Delta v\right] + \ldots \qquad \text{(B6)}$$

Interpretation of Prediction Error

Eq. (B6) is easier to interpret if we first note that the linear regression predicted value for, say, Δs, based on the sample data is $\rho_{\Delta s\Delta y}\frac{\sigma_{\Delta s}}{\sigma_{\Delta y}}\Delta y$.

In other words, if we had to predict the change in Δs given Δy, then historical data would give the linear regression predictor as

$$\hat{\Delta} s = \rho_{\Delta s\Delta y}\frac{\sigma_{\Delta s}}{\sigma_{\Delta y}}\Delta y$$

Hence the prediction (or hedging) error shown in Eq. (B6) can be rewritten as

$$Error \cong \mu\Delta y + D_s(\hat{\Delta} s - \Delta s) + D_v(\hat{\Delta} v - \Delta v) + \ldots \qquad \text{(B7)}$$

Eq. (B7) states that if we use empirical duration, then hedging errors will be due to changes in OAS and other risk factors displaying relationships with changes in *y* that differ from those displayed in the past, and also due to shortening or lengthening of duration over the period. In contrast, Eq. (A9) in Appendix A states that if we use effective duration, then any differences between actual and projected prices will be due to *changes* in OAS, volatilities, and other risk factors.

NOTES

Investment Code (for a six - to 12-month time horizon)

U.S. equities are coded against the S&P 500; Japanese equities are coded against the Tokyo Stock Price Index (TOPIX); equities from other countries are coded against the appropriate local index.

Buy: Expected to outperform the market. **Hold:** Expected to match the market. **Underperform:** Expected to underperform the market.

Liquidity Code (based on 20 prior trading days)

U.S. stocks are coded based on a descending scale of 1-5, with 1 representing the most liquid quintile of the S&P 500 based on the dollar value of shares traded. The quintile ranges are printed each week in the *Equity Research Weekly*.

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